THE

B: Scott

ELEMENTS

OF

MATHEMATICAL ANALYSIS,

ABRIDGED.

FOR THE USE OF STUDENTS.

WITH

NOTES,

DEMONSTRATIVE AND EXPLANATORY.

AND

A SYNOPSIS OF BOOK V. OF EUCLID.

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PREFACE.

IN this ſmall Treatiſe on Mathematical Analyſis, the moſt general Propoſitions ONLY are given; and in the Notes at the bottom of the pages, are pointed out the principal authors to be conſulted by ſtudents. The whole to page 129, was printed in 1777; but owing to the Author's bad health, &c. it was not until 1783, that the ſame with the addition of the quarter ſheet from page 129 to page 133 (which ſhould have come in after Propoſition VII.), was uſed here as a Text Book. As now publiſhed, the Definitions formerly omitted, are placed properly before the Axioms; and in the NOTES concluding the Analyſis, from page 133 to page 148, are given demonſtrations of ſeveral propoſitions, and corrections of ſeveral errata in the text; all other errata of conſequence, being pointed out in the table following this preface.

Several Corollaries to Propoſitions XV. XVII. having been omitted by miſtake in the printing, the reader is referred to the truly learned and judicious Mr Baron Maſeres Diſſertation on the Negative Sign, where the reduction of quadratic and cubic equations, &c. is given at great length, and in the moſt perſpicuous and ſcientific manner.

The ſecond Corollary (54) of Propoſition XXI. being imperfect as PRINTED, may be corrected in manner following, viz. IN EVERY AFFECTED EQUATION, THE COEFFICIENTS OF THE INTERMEDIATE TERMS BETWEEN THE FIRST AND LAST, COMPREHEND ALWAYS diviſors ſimple or compound of the laſt term, in the component parts of their ſums and differences.

The rule of intereſt (115), as now given (207), is general for all rates PER CENT. PER ANNUM. And the rule of intereſt (210), is general alſo for all rates PER CENT. PER ANNUM, perfectly accurate, and more ſimple and eaſy in appplication, than any rule hitherto given.

The preſent Treatiſe is an abridgment of part of a comprehenſive Syſtem of the Elements of Mathematical Analyſis, common and fluxionary, now almoſt finiſhed.

With regard to the Sign of Inequality, here introduced, the diſcriminating letters *r*, *s*, ſhould have been in a ſmaller character, and placed above the angular point of the ſign; or more SPACE ſhould have been taken by the printer, to prevent any ambiguity when the ſymbols following are *r*, *s*, or any powers of the ſame.

ERRATA.

ANALYSIS. Page 11, line 1, read $\left(=\frac{a}{b}\times\frac{c}{1}\right)$.

Page 16, Art. 17, lines 3, 4, read $ax-b=cx+d$, $ay^m+bc=d$.

—— 26, before Art. 30, add, SCHOLIUM III.—Lines 3, 4, dele *when* m *is ſome power of the number* 2.—Lines 5, 7, read binomial.

—— 27, *Note* *, line 1, read Willebrordus.

—— 28, Art. 32, line 3, read z^2 is wanting.—Art. 34, line 3, for $\surd 3q$ read $\frac{2r}{3q}$.

—— 29, line 4, connect the cubic radicals with +.

—— 32, Art. 39, Ex. 2, 4, make firſt term x^3.

—— 34, Ex. 2, line 6, at the end, add, *in whole numbers*.

—— 42, Art. 48, lines 13, 14, inſert laſt values of v, y, viz. 8, 12.

—— 44, Ex. 3, lines 1, 2, read $Q=-1$, $z^3+3z=36$, gives $2B=3$ (47).—Ex. 5, line 1, read 18252.—Ex. 6, line 3, read $v=8$.—Ex. 7, line 3, read $\frac{1}{2}\surd-\frac{1}{3}$.

—— 47, Art. 52, at the end, add, *with their ſigns changed*.

—— 49, Art. 56, line 6, before the colon, add, *taking in the laſt quotient*.—Art. 57, Ex. 2, line 3, read roots 3, 5, —6.—Ex. 3, line 1, read $106x$.

—— 50, Art. 58, Ex. 2, line 2, read 3, 5, —6; and line 3, read —3, —5, 6.—Ex. 3, lines 1, 3, read $106x$.—Art. 61, line 5, read $x+a=y$, or $x=y-a$.

—— 52, Art. 63, lines 7, 8, read $x^m\pm px^{m-1}$, &c. and $y\mp\frac{p}{m}=x$.

—— 75, line 1, prefix 85, Corol. V. and correct the Article Numbers to page 81.

—— 92, Caſe III. in values of n, v, for $2s+d$, read $2s+\frac{1}{4}d$.

—— 93. Caſe VII. in value of s, read $\frac{n-1}{2}$.—Caſe VIII. in values of a, n, read $-d\times\overline{2s-\frac{1}{4}d}$, and for $v=$&c. read $\frac{v\pm\surd\&c.}{d}+\frac{1}{2}$.

—— 111. Note ‡. See Abbot de Molieres in his Mathematical Leſſons, London, 1730.

—— 118. Caſe VI. in value of r, dele —, and in value of s, to the fraction add $v-$. Caſe VIII. read $a=rv-s\times\overline{r-1}$; and hence correct the value of n.

—— 120, 121, lines 5, 16, read $ra\times\frac{1}{r-1}$.—Art. 139, line 1, read *whole* numbers.

—— 123, Art. 141, lines 6, 7, for $2^{n\times1}$ read 2^{n+1}.

—— 131, Art. 157, line 1, read $Q:q=\frac{1}{V}:\frac{1}{v}$.

NOTES. Page 132, Art. 168, laſt line, dele × before $=$.

Page 136, line 20, for Hyp. read Since.

—— 138, Art. 187, line 2, read the roots; and line 3, read term.

SYNOPSIS. Page 4, line laſt but one, read ratio 5 : 7.

Page 13, Prop. D. Corol. line laſt, for greater leſs, read equally leſs.

—— 21, line 15, on the left, read Prop. X.

DEFINITIONS.

DEFINITION I.

MATHEMATICAL ANALYSIS is the ſcience of QUANTITY in general; where by QUANTITY is underſtood whatever is meaſurable, or made up of parts.

DEFINITION II.

QUANTITY is divided into PROPER and IMPROPER. That is called *proper quantity* which can be meaſured by its own *kind*, and that *improper quantity* which cannot be meaſured by its own *kind*, but to which a meaſure may be aſſigned by means of ſome *proper quantity* having connection with it *.

DEFINITION III.

In MATHEMATICAL ANALYSIS all quantities are repreſented by alphabetical letters. And the ALGORITHM OF ARITHMETIC of quantity admits of the ſame operations as that of *numbers*, viz. addition, ſubtraction, multiplication, diviſion, involution, and evolution.

DEFINITION IV.

ADDITION collects or gathers together quantities into one *ſum* or *aggregate*. And the operation of addition is *denoted* by interpoſing between the quantities to be added together the *ſign* of addition (+ *plus*).

DEFINITION V.

SUBTRACTION determines the *difference* of quantities. The quantity from which the ſubtraction is made is called the *minuend*, and that by which ſubtraction is made the *ſubtrahend* or *ablative quantity*. And the operation of ſubtraction is *denoted* by interpoſing between the minuend and ſubtrahend the *ſign* of ſubtraction (— *minus*).

DEFINITION VI.

MULTIPLICATION compounds or repeats one quantity as often as there are UNITS in the ſame, or any other quantity. The quantity ariſing by mul-

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* Vid. Philoſophical Tranſactions, No. 489, for year 1748.

tiplication is called a *product*, the quantities multiplied together being the *factors* of the product; whereof one is termed the *multiplier*, and the other the *multiplicand*. And the operation of multiplication is *denoted* by interpoſing between the factors the *ſign* of multiplication (× *into*); or ſimply by the *juxtapoſition* of the factors.

DEFINITION VII.

Division decompounds or reſolves a product into its two factors or efficient parts, one of them being always aſſigned. The product to be reſolved by diviſion is called the *dividend;* the factor by which diviſion is made, the *diviſor;* and that ariſing by diviſion, the quotient. And the operation of diviſion is *denoted* by interpoſing between the dividend and diviſor the *ſign* of diviſion (÷ *by*); or by placing the dividend *above* a ſtraight line, and the diviſor *under* it; the upper quantity or dividend being then termed the *numerator*, and the lower quantity or diviſor the *denominator* of the fraction or quotient.

COROLLARY I.

A quotient, then, in reference to its diviſor and dividend, is univerſally the quantity, which, multiplied into the diviſor, produces the dividend (Def. VI.).

COROLLARY II.

A fractional expreſſion, in reference to its denominator, or the product of the denominator by any integral quantity, is the numerator of the fraction, or the product of the numerator by the ſame integral quantity *.

DEFINITION VIII.

Involution compounds or repeats one quantity *continually*, as often as there are units in itſelf, and in all ſubſequent products; and theſe products are called POWERS, the generating quantity being the RADIX of the ſcale or *firſt* power. And the operation of involution is *denoted* by placing, at the top of the root upon the right hand, a ſmall numeral figure, expreſſive of the number of repetitions of the radix at any place or term of the ſeries; which numeral figure is emphatically called the INDEX, EXPONENT, or CHARACTERISTIC of the power.

DEFINITION

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* Thus, in reference to 4, 8, 12, 16, 20, 24, 28, &c. $\frac{1}{4}$ is 1, 2, 3, 4, 5, 6, 7, &c. and $\frac{3}{4}$, therefore, 3, 6, 9, 12, 15, 18, 21, &c. and the ſame with regard to any other fraction.

DEFINITION

DEFINITION IX.

Evolution decompounds or reſolves powers into roots, which being the converſe of involution (Def. VIII.), is therefore effected by *continual* diviſion. And the operation of evolution is *denoted* in the Newtonian method by giving to the quantity as an *index unit, divided by the exponent of the particular power* to be reſolved: Or the ſame may be denoted in the common way by placing over the quantity the radical sign having the index of the *particular power* prefixed thereto ($\sqrt[m]{}$).

COROLLARY.

Since multiplication (Def. VI.) and diviſion (Def. VII.) are only repeated addition (Def. IV.) and repeated ſubtraction (Def. V.); and ſince involution (Def. VIII.) and evolution (Def. IX.) are only repeated multiplication and repeated diviſion; therefore addition and subtraction are the two primary or fundamental operations in the algorithm of quantity (Def. III.); and of conſequence the signs of addition and ſubtraction the two *capital ſigns* in the analyſis of quantity (Def. I.).

DEFINITION X.

Simple quantities are thoſe expreſſed by ſingle ſymbols. Compound quantities are thoſe expreſſed by different ſymbols, connected together by the ſigns of addition and ſubtraction: And in every Expreſſion of compound quantities, the connecting *ſign* belongs to the quantity before which it is placed; any quantity being conſidered as *affirmative* or *negative* according as it is affected with the ſign of addition or ſubtraction: A quantity without any ſign being always affirmative.

DEFINITION XI.

A numeral figure, whether integral or fractional, prefixed to any quantity, is called a coefficient or fellow-factor, as *denoting* how often the quantity is reckoned; and the quantity itſelf, with its coefficient prefixed, is called a multiple. A quantity with a fractional coefficient, is ſometimes called submultiple. And when no number is prefixed to any quantity, unity is underſtood as the coefficient.

COROLLARY I.

Hence, aſſuming any quantity A; ſince $A=1\times A$ (Def. VI.) and $\frac{1}{A}\times A=1$ (Def. VII. Corol. I.); that is, ſince A is the ſame multiple of unity as $\frac{1}{A}$ is part or ſubmultiple of unity, the quantities A, $\frac{1}{A}$, are therefore ſaid to be INVERSE OF RECIPROCAL of one another. And hence, in the Newtonian notation of evolution (Def. IX.), the index or exponent of the root is the inverſe or reciprocal of the index or exponent of the power to be evolved.

COROLLARY II.

The coefficient of a fraction, if a whole number, may be prefixed to the numerator of the fraction, and if a fractional number, the numerator of the coefficient may be prefixed to the numerator of the fraction, and the denominator of the coefficient to the denominator of the fraction. For a fraction being an expreſſion of a quotient (Def. VII.) to repeat a fraction according to the units in any number (Def. VI.), is to multiply the number into the numerator, the denominator remaining the ſame: And to take any part of a fraction according to the units in any number, or to divide a fraction by a number, is to multiply the number into the denominator, the numerator remaining the ſame.

DEFINITION XII.

SIMILAR OF LIKE QUANTITIES are thoſe expreſſed by the ſame ſymbol or ſymbols equally repeated; and DISSIMILAR OF UNLIKE QUANTITIES are thoſe expreſſed by different ſymbols, or by the ſame ſymbol or ſymbols differently repeated.

DEFINITION XIII.

A STRAIGHT LINE drawn above any compound quantity, and conſolidating as it were the compound quantity into one maſs, is called a VINCULUM. And by means of the VINCULUM, the ſums, differences, products, quotients, powers, and roots of compound quantities, are *denoted* in the ſame manner as thoſe of ſimple quantities.

DEFINITION XIV.

The EQUALITY of quantities perfectly equivalent to one another, is *denoted* by interpoſing between the equivalent quantities the ſign of equality (=): and every expreſſion of equivalent quantities denoted in this manner, is called an EQUATION. And when there are different quantities connected together by the *ſigns* of addition and ſubtraction, on both ſides of an equation, theſe are called TERMS OF MEMBERS of the equation.

DEFINITION XV.

The PRINCIPAL QUANTITY in an equation is that quantity whereof the VALUE OF VALUES depend upon, or flow from the *values* of all the other quantties quantities combined any how in the terms of the equation. This quantity is denominated the ROOT of the equation; the VARIABLE, the INDETERMINATE, or the UNKNOWN quantity; and is always *denoted* or repreſented by ſome one of the final letters of the alphabet.

DEFINITION XVI.

THOSE QUANTITIES in an equation whereof the VALUES depend not upon the combination and arrangement of the terms, but continue always the *ſame*, let them be combined any how, are called INVARIABLE, or DETERMINATE, or KNOWN, or GIVEN quantities; and theſe are always *denoted* or repreſented by ſome of the *initial letters* of the alphabet.

DEFINITION XVII.

REDUCTION OF AN EQUATION brings the principal quantity (Def. XV.) to poſseſs one ſide of the equation *alone*, and totally diſengaged from other quantities; and the PRINCIPAL QUANTITY is then DETERMINED, as being expounded on the other ſide of the equation by a quantity equivalent thereto, in conſequence of that combination of the INVARIABLE QUANTITIES (Def. XVI.) from which the equation derived its *origin*.

COROLLARY.

Any quantity in an equation may be conſidered as the PRINCIPAL QUANTITY, and may be enunciated by REDUCTION; a value poſſible and conſiſtent in all reſpects being *previouſly* ſubſtituted inſtead of the *principal quantity* wherever it appears in the equation.

DEFINITION XVIII.

RATIO is a mutual relation of two magnitudes of the ſame kind to one another in reſpect of quantity: Or according to Euclid in the Elements *,

RATIO is a certain mutual habitude of two magnitudes of the ſame kind, according to quantity: That is, the mutual compariſon of two homogeneous quantities according to magnitude or quantity, is called RATIO, and is *denoted* by interpoſing between the quantities two points placed vertically (:). The quantities compared together are called the terms of the ratio; the former quantity being denominated the *antecedent*, and the latter the *conſequent:* And the ratio is denominated a ratio of *majority*, or of greater inequality, *equality*, or of equal to equal, and *minority* or of leſs inequality, according as the antecedent is greater, equal, or leſs than the conſequent.

COROLLARY

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* Book V. Definition III.

COROLLARY I.

Since any quantity is equal to *unity* repeated as often as there are units in ſaid quantity (Def. VI.), it therefore follows that any quantity may be conſidered, and hence *denoted* as a *ratio* having unity for a conſequent: And becauſe any quantity may be conſidered as a *unity* of its own kind, it follows alſo that a ratio undergoes no change of VALUE, by taking inſtead of the terms of the ratio, any EQUIMULTIPLES, integral or fractional of the ſame (Def. XI.). Thus in the ratio m A : A, conſidering A as 1, then m A as m (Def. XI.), and the ratio m A : A the ſame with the ratio m : 1; thus alſo, in the ratio m A : n A, conſidering A as 1, then m A as m, n A as n, and the ratio m A : n A the ſame with the ratio m : n. And hence all ratios may be reduced to *equivalent* ratios having unity for a common conſequent, by taking the general expreſſions of the quotients of the antecedents by the conſequents (Def. VII.) as the antecedent.

COROLLARY II.

EQUAL RATIOS obtain between adjacent terms, and UNEQUAL RATIOS between terms that are *not* adjacent in any *regular* ſeries of powers (Def. VIII.) or roots (Def. IX.); and alſo in any ſeries of quantities where all terms of the ſeries are EQUIMULTIPLES of immediately *contiguous* terms, reckoning from either extremity of the ſeries, and taking any quantity, integral or fractional, as the multiplier *. In the ſeries of integral terms increaſing, or fractional terms decreaſing, the product of two aſſumed terms, in reference to the higher aſſumed term, is always a ſubſequent or higher term, and the quotient of the higher aſſumed term, divided by the lower, a preceding or lower term. Whence, ſince any term may be conſidered as a ratio, whereof the conſequent is unity (Corol. I.), it is evident that theſe ratios are *generated* from a multiplication or diviſion of their antecedents; the former operation generating *greater*, and the latter *leſs* ratios or terms; therefore, conſidering every ratio as a quantity *ſui generis*, the former operation in the arithmetic of ratios is called ADDITION, and the latter SUBTRACTION.

COROLLARY

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* 1, 2, 4, 8, 16, 32, 64, 128, 156, &c.
$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256},$ &c.
1, 3, 9, 27, 81, 144, 739, 2187, 6591, &c.
$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \frac{1}{729}, \frac{1}{2187}, \frac{1}{6591},$ &c.

&c. &c.

1, m A, $m\,m$ A, $m\,m\,m$ A, $m\,m\,m\,m$ A, $m\,m\,m\,m\,m$ A, &c.

&c. &c.

COROLLARY III.

Between *equal* quantities there can be *no* ratio, accurately ſpeaking. The idea of equality being that of ſameneſs or identity, and ſo oppoſed to the idea of compariſon, as implying that of diverſity. Hence a ratio of equality is ſaid to be *nothing*, or to have *no* meaſure; and converſely, of a ratio equal to *nothing*, the terms are equal. Hence,

COROLLARY IV,

Aſſuming any quantity and itſelf again, or which is the ſame, aſſuming two equal quantities, and ſtating the ratio in terms, let the *conſequent remain invariably the ſame*. Then if the antecedent be ſuppoſed to increaſe continually, or to flow upward through all degrees of magnitude, the ſeries will comprehend *all* ratios of majority or greater inequality, and theſe *increaſing* continually as the ſeries expands continually: but if the antecedent be ſuppoſed to diminiſh continually, or to flow downwards through all inferior degrees of magnitude, the ſeries will comprehend *all* ratios of minority or leſs inequality, and theſe *increaſing* continually as the ſeries expands downwards continually. Thus the ratio of equality, though properly ſpeaking *no* ratio, is the *common limit* between ratios of majority and ratios of minority: and the *magnitude* of any ratio may be eſtimated by its *diſtance* from the common limit or ratio of equality, a ratio being *greater* the *farther* it is removed from the ratio of equality: And from the *different ways* by which *all* ratios of majority and minority are derived, in the ſeries expanded upwards and downwards indefinitely, it appears that theſe ratios are quite *oppoſite in kind*, or heterogeneous to one another, and ſo incapable of any mutual compariſon as to magnitude or quantity.

COROLLARY V.

Hence, in comparing together ratios as to equality and inequality, if equimultiples of both terms of the ratios, compared together by the alternate conſequents or antecedents, give the ſame ratio or equal ratios, the ratios compared together are equal (Corol. I.); and otherwiſe unequal: And of unequal ratios, that univerſally is the greater ratio which is moſt remote or diſtant from the ratio of equality (Corol. IV.).

Ratios Equal

$$\begin{array}{r} 3:2 \\ 81:54 \end{array}\left\{\begin{array}{l} 3:2=3\times54:2\times54=162:108 \\ 81:54=81\times2:54\times2=162:108 \end{array}\right\}3:2=81:54$$

Ratios Equal

$$\begin{array}{r} 5:7 \\ 115:161 \end{array}\left\{\begin{array}{l} 5:7=5\times115:7\times115=575:805 \\ 115:161=115\times5:161\times5=575:805 \end{array}\right\}5:7=115:161$$

$$\begin{array}{llr} \text{Ratios} & & \text{Unequal} \\ \left.\begin{array}{r} 7:5 \\ 11:9 \end{array}\right\{ & \left.\begin{array}{l} 7:5 = 7\times 9:5\times 9 = 63:45 \\ 11:9 = 11\times 5:9\times 5 = 55:45 \end{array}\right\} & \begin{array}{r} 7:5 > r\ 11:9 \\ 45:45 \cdots\cdots 55:45 \cdots\cdots 63:45 \end{array} \end{array}$$

$$\begin{array}{llr} \text{Ratios} & & \text{Unequal} \\ \left.\begin{array}{r} 5:7 \\ 9:11 \end{array}\right\{ & \left.\begin{array}{l} 5:7 = 5\times 11:7\times 11 = 55:77 \\ 9:11 = 9\times 7:11\times 7 = 63:77 \end{array}\right\} & \begin{array}{r} 5:7 > r\ 9:11 \\ 77:77 \cdots\cdots 63:77 \cdots\cdots 55:77 \end{array} \end{array}$$

SCHOLIUM.

This method of comparing together ratios of minority as to inequality, or as greater or leſs, differs from that given by **Euclid**, VI! Def. e. v. And from this definition of **Euclid**, as applied to ratios of minority, it might be inferred, admitting, or ſuppoſing negative quantities, that —2 is greater than —18 for example, though in truth but the 9th part of the ſame. **Euclid**, it muſt be obſerved, conſiders a ratio of minority as greater the nearer it is to, and not the farther it is from, the ratio of equality; it being obvious that 3, for example, has more magnitude when compared with 5, than it has when compared with 17; and that, therefore, the ratio 3 : 5 conſidered abſtractly, is greater than the ratio 3 : 17: But, conſidering the ratios 3 : 5, 3 : 17, as what they really are, namely, as ratios of minority, the ratio 3 : 5 is evidently a leſs ratio of *minority* than the ratio 3 : 17, and accordingly has a leſs *logarithmic* meaſure.

DEFINITION XIX.

The INEQUALITY of quantities being either in exceſs or defect, is *denoted* accordingly, by interpoſing between the unequal quantities the proper ſign of inequality (> *r greater* or > *s leſs*).

AXIOMS:

AXIOM I.

Every quantity is equal to itſelf.

AXIOM II.

The remainder of a quantity ſubtracted from itſelf, or from an equal quantity, is nothing.

AXIOM III.

Any whole is equal to all its parts.

AXIOM IV.

A quantity expreſſed one way, is equal to itſelf expreſſed any other way.

AXIOM V.

Quantities which are equal to one and the ſame quantity, or to equal quantities, are themſelves equal.

AXIOM VI.

Equimultiples of the ſame, or of equal quantities, are themſelves equal.

AXIOM VII.

If equal quantities be added to equal quantities, the ſums will be equal.

AXIOM VIII.

If equal quantities be ſubtracted from equal quantities, the remainders will be equal.

AXIOM IX.

If equal quantities be multiplied by equal quantities, the products will be equal.

AXIOM X.

If equal quantities be divided by equal quantities, the quotients will be equal.

AXIOM XI.

If equal quantities be equally involved or evolved, the powers or roots will be equal.

MATHEMATICAL ANALYSIS.

SECTION I. *The Algorithm of Quantity.*

1. $+$ *Plus*. The ſign of addition. $A+B$, ſum of A and B.

$-$ *Minus*. ——— ſubtraction. $A-B$, is B ſubtracted from A.

$\times$ *Multiplication.* $A\times B$ or AB, the product of A by B.

$\div$ *Diviſion.* $A\div B$, or $\frac{A}{B}$, the quotient of A divided by B.

$=$ *Equality*. Thus $A=B$, an equation.

: *Ratio*. Thus $A:B$, denotes the ratio of A to B.

$\overline{A+B}+\overline{C-D}$. The ſum of the compound quantities $A+B$ and $C-D$.

$\overline{A+B}-\overline{C-D}$. The difference.

$\overline{A+B}\times\overline{C-D}$. The product.

$$\left.\begin{array}{r}\overline{A+B}\div\overline{C-D}\\ \text{Or, }\dfrac{A+B}{\overline{C-D}}\end{array}\right\}\text{ The quotient.}$$

Involution, or continual Multiplication. Raiſing of Powers.

$$\begin{array}{r}\text{Powers}\\ \text{of A}\end{array}\left\{\begin{array}{lllll}A, & AA, & AAA, & AAAA, & AAAAA,\ \&c.\\ & & & \text{Or,} & \\ A, & A^2, & A^3. & A^4, & A^5,\ \&c.\end{array}\right.$$

$$\begin{array}{r}\text{Powers of}\\ A\pm B.\end{array}\left\{\ \overline{A\pm B}^{\,1},\ \overline{A\pm B}^{\,2},\ \overline{A\pm B}^{\,3},\ \overline{A\pm B}^{\,4},\ \&c.\right.$$

Evolution, or continual diviſion. Extraction of Roots.

Roots of A $\left\{ \begin{array}{l} \sqrt{A},\ \sqrt[3]{A},\ \sqrt[4]{A},\ \sqrt[5]{A},\ \textit{\&c.}\ \text{or,} \\ A^{\frac{1}{2}},\ A^{\frac{1}{3}},\ A^{\frac{1}{4}},\ A^{\frac{1}{5}},\ \textit{\&c.} \end{array} \right.$

Roots of $\overline{A \pm B}$. $\left\{ \begin{array}{l} \sqrt{A \pm B},\ \sqrt[3]{A \pm B},\ \sqrt[4]{A \pm B},\ \textit{\&c.} \\ \text{Or,} \\ \overline{A \pm B}|^{\frac{1}{2}},\ \overline{A \pm B}|^{\frac{1}{3}},\ \overline{A \pm B}|^{\frac{1}{4}},\ \textit{\&c.} \end{array} \right.$

2. *The uſe of the* Vinculum, *in expreſſing the ſums, differences, products, and quotients of compound quantities.*

Suppoſe $A = 12$, $B = 5$, $C = 20$, and $D = 15$,

$$\overline{A+B} + \overline{C-D} = \overline{12+5} + \overline{20-15} = 17+5 = 22.$$

$$\overline{A-B} + \overline{C-D} = \overline{12+5} + \overline{20-15} = 7+5 = 12.$$

$$\overline{C+D} - \overline{A+B} = \overline{20+15} - \overline{12+5} = 35-17 = 18.$$

$$\overline{C+D} - A+B = \overline{20+15} - 12+5 = 40-12 = 28.$$

$$\overline{C+D} - \overline{A-B} = \overline{20+15} - \overline{12-5} = 35- 7 = 28.$$

$$\overline{C+D} - A-B = \overline{20+15} - 12-5 = 35-17 = 18.$$

$$\overline{A+B} \times C = \overline{12+5} \times 20 = 17 \times 20 = 340.$$

$$A+B \times C = 12+5 \times 20 = 12+100 = 112.$$

$$\overline{A+B} \times \overline{C-D} = \overline{12+5} \times \overline{20-15} = 17 \times 5 = 85.$$

$$\overline{A+B} \times C-D = \overline{12+5} \times 20-15 = 340-15 = 325.$$

$$\overline{C+D} \div \overline{A-B} = \overline{20+15} \div \overline{12-5} = 35 \div 7 = 5.$$

$$\text{Or, } \frac{C+D}{A-B} = \frac{20+15}{12-\ 5} = \frac{35}{7} = 5.$$

$$\overline{C+D} \div \overline{B+A} = \overline{20+15} \div \overline{5+12} = \frac{35}{17} = 2\frac{1}{17}.$$

$$\overline{C+D} \div B+A = \overline{20+15} \div 5+12 = 7+12 = 19.$$

Note. This ſign $\backsim$ denotes univerſally the difference of any two quantities.

PROPOSITION

PROPOSITION I.

3. The ſum of any number of quantities is *expreſſed*, by connecting quantities with their proper ſigns, and *uniting* ſuch as can be *united*. But, in the addition of ratios, the product of the antecedents gives the antecedent of the ſum, and that of the conſequents, the conſequent of the ſum.

Ex. 1.

$$4ab + 2b^2 - 2abc - 5a^2b^2 + m + \frac{7a}{b} - \frac{4c^2}{a} + \frac{a^2}{c}$$

$$8ab - 2b^2 - 4abc + 3a^2b^2 - n + \frac{12a}{b} + \frac{3c^2}{a} + \frac{bx}{c}$$

$$12ab - 6abc - 2a^2b^2 + m - n + \frac{19a}{b} - \frac{c^2}{a} + \frac{a^2 + bx}{c}.$$

Ex. 2.

$$8a^2b + \frac{10a^2}{b^2} + 12\sqrt{\frac{bc}{a}} - 3\sqrt{ab - x^2} + \sqrt[3]{a^5b} + \frac{a}{b}$$

$$3a^2b - \frac{4a^2}{b^2} - 7\sqrt{\frac{bc}{a}} + 12\sqrt{ab - x^2} - \sqrt[3]{a^5b} + \frac{c}{d}$$

$$11a^2b + \frac{6a^2}{b^2} + 5\sqrt{\frac{bc}{a}} + 9\sqrt{ab - x^2} + \frac{a}{b} + \frac{c}{d}.$$

Ex 3.

$$\overline{A:B} + \overline{C:D} = AC:BD.$$

Ex. 4.

$$\overline{A:B} + \overline{B:C}\,(= AB:BC) = A:C.$$

PROPOSITION II.

4. Subtraction in general takes place by connecting the quantities together, as in addition, ſuppoſing the ſigns of the ſubtrahend changed. But, in the ſubtraction of ratios, the terms of the ſubtrahend, or ablative ratio, muſt be inverted.

Ex. 1.

$$\begin{array}{l}
12ab - 6abc - 2a^2b^2 + m - n + \frac{19a}{b} - \frac{c^2}{a} + \frac{a^2}{c} + \frac{bx}{c} \\
4ab - 2abc - 5a^2b^2 + m \qquad + \frac{7a}{b} - \frac{4c^2}{a} + \frac{a^2}{c} \\
\hline
8ab - 4abc + 3a^2b^2 - n + \frac{12a}{b} + \frac{3c^2}{a} \qquad + \frac{bx}{c}
\end{array}$$

Ex. 2.

$$\begin{array}{l}
11a^2b + \frac{6a^2}{b^2} + 5\sqrt{\frac{bc}{a}} + 9\sqrt{ab - x^2} + \frac{a}{b} \qquad + \frac{c}{d} \\
3a^2b - \frac{4a^2}{b^2} - 7\sqrt{\frac{bc}{a}} + 12\sqrt{ab - x^2} - \sqrt[3]{a^5b} + \frac{c}{d} \\
\hline
8a^2b + \frac{10a^2}{b^2} + 12\sqrt{\frac{bc}{a}} - 3\sqrt{ab - x^2} + \sqrt[3]{a^5b} + \frac{a}{b}.
\end{array}$$

Ex. 3.

$$\overline{A:B} - \overline{C:D}\ (= \overline{A:B} + \overline{D:C}) = AD : BC.$$

Ex. 4.

$$\overline{A:B} - \overline{B:C}\ (= \overline{A:B} + \overline{C:B}) = AC : B^2.$$

Proposition III.

5. Multiplication connects the products of the factors with the ſigns + or — according as the factors have like or unlike ſigns; every term of the multiplier being drawn into each term of the multiplicand ſucceſſively.

Ex. 1. $\overline{a+b} \times \overline{a+b} = a^2 + 2ab + b^2$. *Ex.* 2. $\overline{a+b} \times \overline{a-b} = a^2 - b^2$.

$$\begin{array}{r}
a + b \\
a + b \\
\hline
a^2 + ab \phantom{{}+b^2} \\
ab + b^2 \\
\hline
a^2 + 2ab + b^2
\end{array}
\qquad\qquad
\begin{array}{r}
a + b \\
a - b \\
\hline
a^2 + ab \phantom{{}-b^2} \\
-ab - b^2 \\
\hline
a^2 - b^2.
\end{array}$$

Ex. 3. $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$. Ex. 4. $\frac{a}{b}\times c\left(=\frac{a}{b}+\frac{c}{1}\right)=\frac{ac}{b}$.

Ex. 5. $\sqrt[m]{a}\times\sqrt[m]{b}=\sqrt[m]{ab}$.

Ex. 6. $\overline{3a-2b+2c}\times\overline{2a-4b+5c}=$
$6a^2-16ab+19ac+8b^2-18bc+10c^2$

$$\begin{array}{l}
3a-2b+2c \\
\underline{2a-4b+5c} \\
6a^2-4ab+4ac \\
\quad -12ab \qquad\quad +8b^2-8bc \\
\qquad\qquad +15ac \qquad\quad -10bc+10c^2 \\
\hline
6a^2-16ab+19ac+8b^2-18bc+10c^2.
\end{array}$$

Ex. 7.

Let m represent any number, then $m\times\overline{A:B}=A^m:B^m$.

6. Corollary. Powers of the ſame quantity may be multiplied by adding the exponents of the factors.

Thus: $a^m\times a^n=a^{m+n}$.

Proposition IV.

7. In diviſion, the quotient quantity is determined by comparing the dividend with the diviſor, expunging all quantities that are common efficients in both, and making the reſult affirmative or negative, according as the ſigns of the diviſor and dividend are like or unlike. Or, the quotient is univerſally that quantity, which, multiplied into the diviſor, produces the dividend.

Ex. 1.

$$\begin{array}{l}
a+b)\ a^2+2ab+b^2\ (a+b=\dfrac{a^2+2ab+b^2}{a+b} \\
\qquad\underline{a^2+ab} \\
\qquad\qquad ab+b^2 \\
\qquad\qquad \underline{ab+b^2.} \\
\qquad\qquad\ \cdot\quad\cdot
\end{array}$$

Ex. 2.
$$\begin{array}{l}
a-b)\ a^2-b^2\ (a+b=\dfrac{a^2-b^2}{a-b} \\
\qquad\underline{a^2-ab} \\
\qquad\qquad ab-b^2 \\
\qquad\qquad \underline{ab-b^2.} \\
\qquad\qquad\ \cdot\quad\cdot
\end{array}$$

Ex. 3. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$. *Ex.* 4. $\frac{a}{b} \div c = \frac{a}{bc}$.

Ex. 5. $\sqrt[m]{\overline{ab}} \div \sqrt[m]{\overline{b}} = \frac{\sqrt[m]{\overline{ab}}}{\sqrt[m]{b}} \left(= \frac{\sqrt[m]{\overline{a}} \times \sqrt[m]{\overline{b}}}{\sqrt[m]{b}}\right) = \sqrt[m]{a}$.

Ex. 6.

$$\begin{array}{l} 3a-2b+2c)6a^2-16ab+19ac+8b^2-18bc+10c^2(2a-4b+5c \\ \qquad\qquad\quad \underline{6a^2-\ 4ab+\ 4ac} \\ \qquad\qquad\qquad -12ab+15ac+8b^2-18bc \\ \qquad\qquad\qquad \underline{-12ab \qquad\qquad +8b^2-\ 8bc} \\ \qquad\qquad\qquad\qquad\qquad 15ac \qquad -0bc \qquad +10c^2 \\ \qquad\qquad\qquad\qquad\qquad \underline{15ac \qquad -10bc \qquad +10c^2} \end{array}$$

Ex. 7. Let n repreſent any number, then $\overline{A:B} \div n = A^{\frac{1}{n}} . B^{\frac{1}{n}} = \sqrt[n]{\overline{A}} : \sqrt[n]{\overline{B}}$.

8. Corollary. Powers of the ſame quantity may be divided by ſubtracting the exponent of the diviſor from that of the dividend. Thus: $a^m \div a^n = a^{m-n}$.

Proposition V.

9. Involution in general takes place, by multiplying the exponent of the quantity to be involved by that of the power required. (5, 6.)

Thus: $\overline{a}|^{n}$ or a^m raiſed to the n^{th} power, is a^{mn}.

The involving of compound quantities will appear from the following table:

$$\begin{array}{l} \overline{a \pm b}|^{1} = a \pm b \\ \overline{a \pm b}|^{2} = a^2 \pm 2ab + b^2 \\ \overline{a \pm b}|^{3} = a^3 \pm 3a^2b + 3ab^2 \pm b^3 \\ \overline{a \pm b}|^{4} = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4 \\ \overline{a \pm b}|^{5} = a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5, \end{array}$$

&c. &c.

10. Corollary. Hence, ſuppoſing m to repreſent the exponent of any power, then the terms of $\overline{a \pm b}|^{m}$, without their coefficients, will be as in the following ſeries, *viz.*

$a^m,\ a^{m-1}b,\ a^{m-2}b^2,\ a^{m-3}b^3,\ a^{m-4}b^4$, &c.

And as for the coefficients, they will always be determined from this general product, *viz.*

$$1 \times m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}, \ \&c.$$

where, for the coefficient of any term, we muſt take the product of as many factors.

PROPOSITION VI.

11. Evolution in general takes place, by dividing the exponent of the quantity to be evolved by that of the root required, (7, 8).

Thus: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$; and $\sqrt[n]{\overline{a \pm b}^m} = \overline{a \pm b}|^{\frac{m}{n}}$.

Hence, and from the laſt propoſition, may be derived the following rule for the evolution of compound quantities:

Having found the firſt term of the root by this propoſition, ſubtract its proper power from the quantity given to be evolved, and bring down the ſecond term for a dividend; involve the root to the next inferior power to that given, and multiply this power by the exponent of the given power for a diviſor; annex the quotient to the root, and involve the whole to the given power for a ſubtrahend; to the remainder bring down the term next to theſe affected by this ſubtraction for a new dividend; find the diviſor as before, and proceed in this manner until all the terms of the quantity given to be evolved, are exhauſted.

In applying this rule to numbers, let the numbers be diſtinguiſhed into periods, by pointing according to the exponent of the power; and to obtain the decimal places in the root, to every remainder adjoin as many ciphers as the exponent contains units.

Square Root.

Ex. 1.

$$a^2 + 2ab + b^2 \,(a + b$$

$$a^2 \cdot$$

$$2 \times a = 2a)\ \overline{2ab}\,(b$$

$$\overline{a + b}^2 = \underline{a^2 + 2ab + b^2}.$$

Ex. 2.

$$a^2 + 2ab + b^2 - 2ac - 2bc + c^2 \,(a + b - c$$

$$a^2$$

$$2 \times a = 2a)\ \overline{2ab}\,(b$$

$$\overline{a + b}^2 = \underline{a^2 + 2ab + b^2}$$

$$2 \times a = 2a)\ -2ac\,(-c$$

$$\overline{a + b - c}^2 = \underline{a^2 + 2ab + b^2 - 2ac - 2bc + c^2}$$

Cube Root.

Ex. 3. $a^3 - 3a^2b + 3ab^2 - b^3\,(a-b$

$$\begin{array}{l} \quad a^3 \\ 3 \times a^2 = 3a^2)\,\overline{-3a^2b}\,(-b \\ \overline{a-b}^3 = \underline{a^3 - 3a^2b + 3ab^2 - b^3} \end{array}$$

Surſolid Root.

Ex. 4. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5\,(a-b$

$$\begin{array}{l} \quad a^5 \\ 5 \times a^4 = 5a^4)\,\overline{-5a^4b}\,(-b \\ \qquad \underline{a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5} \end{array}$$

Ex. 5. $\dot{1}5\dot{6}.2\dot{5}\,(12.5.$ Square Root.

$$\begin{array}{l} \quad 1 \\ 1 \times 2 = 2)\,\overline{5}\,(2 \\ \overline{12}^2 = 144 \\ 12 \times 2 = 24)\,\overline{122}\,(5 \\ \overline{12.5}^2 = \underline{156.25} \end{array}$$

Ex. 6. $1\dot{3}31\dot{2}05\dot{3}\,(237$ Cube Root.

$$\begin{array}{l} \quad 8 \\ 2^2 \times 3 = 12)\,\overline{53}\,(3 \\ \overline{23}^3 = 12167 \\ \overline{23}^2 \times 3 = 1587)\,\overline{11450}\,(7 \\ \overline{237}^3 = \underline{13312053} \end{array}$$

Ex. 7. $4\dot{8}575\dot{3}241\dot{6}\,(264$ Fourth Root.

$$\begin{array}{l} \quad 16 \\ 2^3 \times 4 = 32)\,\overline{325}\,(6 \\ \qquad 456976 \\ \overline{26}^3 \times 4 = 70304)\,\overline{287772}\,(4 \\ \qquad \underline{4857532416} \end{array}$$

12. Corollary. Since $\overline{a \pm b}^2 = a^2 \pm 2ab + b^2$, where $2ab$, the middle term, is twice the product of the ſquare roots of the two extremes, it is evident that any quantity having theſe properties, will be a perfect ſquare, and therefore the ſquare roots of the extreme terms connected together with the ſign of the middle term, will give the root required.

$$\left.\begin{array}{l} x^2 \pm ax + \frac{1}{4}a^2 \\ 100 + 40 + 4 \\ 100 + 10 + \frac{1}{4} \\ 49 + 7 + \frac{1}{4} \\ 36 + 3 + \frac{1}{16} \\ 16 + 6 + \frac{9}{16} \end{array}\right\} \begin{array}{c} \text{Square} \\ \text{Root.} \end{array} \left\{\begin{array}{l} x \pm \frac{1}{2}a \\ 10 + 2 = 12 \\ 10 + \frac{1}{2} = 10\frac{1}{2} \\ 7 + \frac{1}{2} = 7\frac{1}{2} \\ 6 + \frac{1}{4} = 6\frac{1}{4} \\ 4 + \frac{3}{4} = 4\frac{3}{4} \end{array}\right.$$

PROPOSITION VII.

13. When two ratios are equal, the four terms are ſaid to be *proportional*, or to compoſe an ANALOGY: the two antecedent and conſequent terms are called *homologous* quantities; and the terms of the ſame ratio, *analogous* quantities. And in any analogy, the product of the extreme terms is equal to that of the middle terms; and the analogous terms combined any how by addition and ſubtraction, and ordered alike, are always proportional or analogous.

Suppoſe the Ratio $A:B=C:D$; then $AD=BC$.

And,

$$\text{The Ratio}\left\{\begin{array}{l}A+B:A=C+D:C\\A+B:B=C+D:D\end{array}\right\}\text{Compoſition.}$$

$$\text{The Ratio}\left\{\begin{array}{l}A\backsim B:A=C\backsim D:C\\A\backsim B:B=C\backsim D:D\end{array}\right\}\text{Diviſion.}$$

$$\text{The Ratio}\left\{\begin{array}{l}A:A\underset{\backsim}{+}B=C:C\underset{\backsim}{+}D\\B:A\underset{\backsim}{+}B=D:C\underset{\backsim}{+}D\end{array}\right\}\text{Converſion.}$$

The Ratio $A+B:A\backsim B=C+D:C\backsim D$ Mixtim.

14. Corol. 1. In any analogy the four terms may be taken reciprocally; and this interchanging of the antecedents and conſequents, is called INVERSION: Alſo, when the terms are *all of the ſame kind*, the antecedents and conſequents may be compared together; and this is called ALTERNATION or PERMUTATION.

Thus:

Suppoſe the Ratio $A:B=C:D$;
Then the Ratio, $A:C=B:D$, Alternation;
And the Ratio, $B:A=D:C$, Inverſion.

Moreover, the terms of any one of the equal ratios, and the ſums or differences of the *homologous* terms, are always analogous or proportional. Thus:

Suppoſe the Ratio $A:B=C:D$;
Then the Ratio, $A:B=A+C:B+D$, Syllepſis;
And the Ratio, $A:B=A\backsim C:B\backsim D$, Dialepſis.

15. Corol. 2. In any ſeries of equal ratios, one antecedent has the ſame ratio to its conſequent, as the ſum of all the antecedents to the ſum of all the conſequents.

Suppoſe the Ratio $A:B=C:D=E:F=G:H$, &c.
Then the Ratio $A:B=A+C+E+G+$, &c.$:B+D+F+H+$, &c.

16. Corol. 3. Since, when any four quantities are analogous or proportional, the product of the extreme terms is always equal to that of the middle terms (13), therefore any equation may be converted into an analogy, by taking the factors which compoſe one ſide, and making them the two extremes, and the factors which compoſe the other ſide of the equation, and making them the two means.

Thus: Suppoſe $A \times D = B \times C$;
Then, the Ratio $A : B = C : D$, or $D : C = B : A$.
The Ratio $A : C = B : D$, or $D : B = C : A$.
The Ratio $B : A = D : C$, or $C : D = A : B$.

Alſo, if it is $B = \sqrt{AC}$, or $B^2 = AC$; then
The Ratio $A : B = B : C$.
The Ratio $B : A = C : B$.
And the Ratio $C : B = B : A$.

For the *Algorithm* of quantity, ſee Sir Iſaac Newton's Arithmetica Univerſalis, s' Graveſande's Algebra, Maclaurin's Algebra, and Simpſon's Algebra. And for the doctrine of ratios, ſee Book V. of Euclid's Elements, and Saunderſon's Algebra, or the Abridgement of the ſame.

SECTION II. *The Reduction of Equations.*

17. A ſimple equation is that which involves either one variable quantity, or a ſingle power of one variable quantity combined any how with given quantities: As the equations $ax - b = cx + d$, and $ay^m + bc = d$. And thoſe equations are called *compound*, which involve more than one variable quantity together with given quantities; ſuch are theſe equations, $x + y = a$, and $x = by$.

18. An affected equation is that which involves different powers of the ſame variable quantity combined any how with given quantities, and is denominated according to the higheſt power of the variable qauntity. Thus, theſe equations are called *quadratics, viz.*

Caſe I. $x^2 + px = rg$. Caſe II. $x^2 - px = rg$. Caſe III. $px - x^2 = rg$.

The following are termed Cubics, *viz.*

$x^3 \pm px^2 \pm qx = r$, $x^3 \pm px^2 = r$, $x^3 \pm px = r$, $px - x^3 = r$, &c. &c.

The following Biquadratics, *viz.*

$x^4 \pm px^3 \pm qx^2 \pm rx = s$, $x^4 \pm px^3$, &c. $= s$, px, &c. $- x^4 = s$, &c. &c.

And ſo on always according to the higheſt power of the variable quantity.

PROPOSITION VIII.

19. Any term of an equation may be *tranſpoſed* from one ſide to the other, by changing its ſign, (Ax. VII, VIII.)

Ex. 1. Suppoſe $4x = 30 + 3x$; then $(4x - 3x =)\ x = 30$.

Ex. 2. Suppoſe $x \pm b = a$; then $x = a \mp b$.

Ex. 3. Suppoſe $2x + b = a + x$; then $x = a - b$.

PROPOSITION IX.

20. Any ſimple equation, that does not involve fractional quantities, may be reduced by tranſpoſing all the given quantities to one ſide (19), and then dividing all the terms by the *coefficient* of the unknown quantity, (Ax. X.)

Ex. 1. Suppoſe $3x + 20 = 44$; then $3x = 24$ (19) and $x = \frac{24}{3} = 8$.

Ex. 2. Suppoſe $ax - b = cx + d$; then $(ax - cx =)\ \overline{a - c} \times x = b + d$ (19,) and $x = \frac{b + d}{a - c}$.

Ex. 3. Suppoſe $2ac + ab - ax = 3ac + 2ax - 5ab - dx$.

Ex. 4. Suppoſe $2ax + 5ab - 3bd = 2ab - 5ax + 7bd - ac - dx$.

PROPOSITION X.

21. Any ſimple equation, involving fractional quantities, may be reduced, by multiplying all the terms by *each* denominator ſucceſſively (Ax. IX.) Or, by bringing each ſide of the equation to

a complete

a compleat fraction, reducing theſe fractions to a common denominator, and then caſting off the common denominator, (Ax. IX.)

Ex. 1. Suppoſe $\frac{x}{5}+4=10$; then multiplying all the terms by 5, we have $x+20=50$, and therefore $x\,(=50-20)=30$ (19).

Or,

Since $\frac{x}{5}+4=10$, therefore $\frac{x}{5}+\frac{20}{5}=\frac{50}{5}$, and $x=30$, as before.

Ex. 2. Suppoſe $\frac{x}{5}+\frac{x}{3}=x-7$; then multiplying all the terms by 5, we have $x+\frac{5x}{3}=5x-35$, and therefore all the terms of this equation being multiplied by 3, it is $3x+5x=15x-105$, or, $7x=105$ (19), and therefore $x=\frac{105}{7}=15$ (20).

Or,

Since $\frac{x}{5}+\frac{x}{3}=x-7$, therefore reducing all the terms to a common denominator, $\frac{3x+5x}{15}=\frac{15x-105}{15}$, which gives $x=15$, as before.

Ex. 3. Suppoſe $\frac{2x}{3}+4=\frac{7x}{12}+9$.

Ex. 4. Suppoſe $56-\frac{3x}{4}=48-\frac{5x}{8}$.

Ex. 5. Suppoſe $\frac{ax}{x-b}=\frac{cx}{x-d}$.

Ex. 6. Suppoſe $\frac{cx}{2a}-\frac{ac}{2b}=x-\frac{4ad}{3c}$.

Ex. 7. Suppoſe $\frac{2ab^2x}{3c^2d}+\frac{5ac^2}{b^2}=\frac{6cd^2}{a^2}-3x$.

Proposition XI.

22. A ſimple equation, involving any power or root of an unknown quantity, combined any how with given quantities, may be reduced, by bringing the power or radical quantity to poſſeſs one ſide of the equation *alone* (19), and then evolving, or involving, according to the exponent of the power or radical quantity, (Ax. XI.)

Ex. 1. Suppoſe $x^2 = 144$; then $x = \sqrt{144} = 12$.

Ex. 2. Suppoſe $\frac{x^2 - 12}{3} = \frac{x^2 - 4}{4}$; then $x^2 = 36$, (21, 19), and therefore $x = \sqrt{36} = 6$.

Ex. 3. Suppoſe $\frac{5x^2}{16} - 8 = 12$; then $x = 8$.

Ex. 4. Suppoſe $ax^2 + bc = d$.

Ex 5. Suppoſe $ax^m - bc = cd + dx^m$.

Ex. 6. Suppoſe $\sqrt{4x + 16} = 12$; then by involution $4x + 16 = 144$ (8), and therefore $x = 32$ (19, 21).

Ex. 7. Suppoſe $\frac{\sqrt{5x}}{3} + 12 = 17$; then $x = 45$.

Ex. 8. Suppoſe $\sqrt{a + x} = b + \sqrt{x}$.

Ex. 9. Suppoſe $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$.

Ex. 10. Suppoſe $\sqrt[3]{x^3 - a^3} = x - c$.

Proposition XII.

23. Compound equations may be reduced by taking *values* of the *ſame* unknown quantity from different equations, and then equating theſe values, (Ax. V.) An equation will ariſe involving only one unknown quantity, ſuppoſing there were equations given for *each* unknown quantity.

Or,

Compound equations may be reduced by deriving values of the ſame unknown quantity from *ſome* of the equations, and ſubſtituting theſe values, inſtead of the quantities themſelves, in all the *other* equations, (Ax. IV.)

Ex. 1.

$$\text{Suppoſe} \left\{\begin{array}{l} x+y=s \\ x-y=d \end{array}\right\} \begin{array}{l} \ldots\ldots x=s-y \\ \ldots\ldots x=d+y \end{array} \Big\{ 2y=s-d.$$

Thus: $y=\dfrac{s-d}{2}$, and therefore $x\,(=s-y=y+d)=\dfrac{s+d}{2}$.

Or,

$$\begin{array}{ll} x+y=s \ldots\ldots & y=s-x \\ x-y=d \ldots\ldots\ldots\ldots\ldots\ldots & x-\overline{s-x}=d \end{array}$$

Whence $x=\dfrac{s+d}{2}$, as before.

Ex. 2.

$$\text{Suppoſe} \left\{\begin{array}{l} x+y=a \\ x+z=b \\ y+z=c \end{array}\right\} \begin{array}{l} \ldots\ldots x=a-y \\ \ldots\ldots x=b-z \\ \ldots\ldots\ldots\ldots\ldots \end{array} \left\{\begin{array}{l} a-y=b-z \\ \\ y+z=c. \end{array}\right.$$

Thus: $z=\dfrac{c-a+b}{2}$.

And therefore $y\,(=a-b+z)=\dfrac{a-b+c}{2}$.

$$x\,(=a-y)=\frac{a-c+b}{2}.$$

Or,

$$\begin{array}{ll} x+y=a \ \ldots\ldots & y=a-x \\ x+z=b \ \ldots\ldots & z=b-x \\ y+z=c \ \ldots\ldots\ldots\ldots\ldots\ldots & \overline{a-x}+\overline{b-x}=e \end{array}$$

Whence $2x=a-c+b$, as before.

Ex. 3.

$$\text{Suppoſe} \left\{\begin{array}{l} x+y+z=a \ldots\ldots\ldots\ldots x=a-y-z \\ x+y+v=b \ldots\ldots\ldots\ldots x=b-y-v \\ x+z+v=c \ldots\ldots\ldots\ldots x=c-z-v. \\ y+z+v=d. \end{array}\right.$$

$$\begin{array}{l} \hline a-z=b-v \ldots\ldots\ldots\ldots z=a-b+v \\ a-y=c-v. \\ y+z+v=d \ldots\ldots\ldots\ldots z=d-y-v \\ \hline v-b+a=d-y-v \ldots\ldots y=d-a+b-2v \\ a-y=c-v \ldots\ldots\ldots\ldots y=a-c+v \\ \hline v-c+a=d-a+b-2v. \end{array}$$

Thus . . . . $v = \dfrac{b+c+d-2a}{3}$

Whence $y\ (= a-c+v) = \dfrac{a+b+d-2c}{3}$

$z\ (= a-b+v) = \dfrac{a+c+d-2b}{3}$

$x\ (= a-y-z) = \dfrac{a+b+c-2d}{3}$

Or,

$$\begin{array}{ll}
x+y+z=a & \ldots\ldots\ x=a-y-z \\
x+y+v=b & \\
x+z+v=c & \\
y+z+v=d & \\
\hline
a-z+v=b & \ldots\ldots\ldots\ z=a-b+v \\
a-y+v=c & \\
y+v+z=d & \\
\hline
a-y+v=c & \ldots\ldots\ldots\ y=a-c+v \\
y+a+2v-b=d & \\
\hline
3v+2a-c-b=d, & \text{as before.}
\end{array}$$

Ex. 4. Suppoſe $\begin{cases} 4x-5y=2 \\ 6x-7y=4. \end{cases}$

Ex. 5. Suppoſe $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}=32 \\ \dfrac{x}{4}+\dfrac{y}{5}=18. \end{cases}$

Ex. 6. Suppoſe $\begin{cases} ax+by=m \\ cx+dy=n \end{cases}$

Ex. 7. Suppoſe $\begin{cases} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p. \end{cases}$

PROPOSITION XIII.

24. In the reduction of compound equations, if the quantity to be *exterminated* is of different dimenſions in the different equations, then, from the equations in which it is found, others

muſt be derived by comparing and equating the different values of its higheſt power, (Ax. V.) And when the higheſt power is not the ſame in two equations, one of the equations muſt be multiplied by the quantity to be *exterminated*, or ſome power of it, (Ax. IX.)

Ex. 1.

Suppoſe $\begin{cases} x^2 - y^2 = d \;.\;.\;.\;.\;.\;.\; x^2 = d + y^2 \\ x + y = s \;.\;.\;.\;.\;.\;.\; x^2 = \overline{s-y}|^2. \end{cases}$

Thus $y = \dfrac{s^2 - d}{2s} = \tfrac{1}{2}s - \tfrac{1}{2}\,\dfrac{d}{s}.$

And $x\,(= s - y) = \tfrac{1}{2}s + \tfrac{1}{2}\,\dfrac{d}{s}.$

Ex. 2.

Suppoſe $\begin{cases} x + y + \dfrac{y^2}{x} = a \;.\;.\;.\;.\; \overline{x + \dfrac{y^2}{x}}\Big|^2 = \overline{a-y}|^2 \\ x^2 + y^2 + \dfrac{y^4}{x^2} = b \end{cases}$

Thus: $y = \dfrac{a^2 - b}{2a} = \tfrac{1}{2}a - \tfrac{1}{2}\,\dfrac{b}{a}.$

Ex. 3.

Suppoſe $\begin{cases} 2x + z = a \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; 4x^2 = a^2 - 2az + z^2 \\ \left.\begin{array}{l} 2x^2 + 2y^2 + z^2 = b \\ x^2 - y^2 = z^2 \end{array}\right\} \left.\begin{array}{l} 3x^2 + y^2 = b \\ x^2 - y^2 = z^2 \end{array}\right\} 4x^2 = b + z^2. \end{cases}$

Thus: $z\,\left(= \dfrac{a^2 - b}{2a}\right) = \tfrac{1}{2}a - \tfrac{1}{2}\,\dfrac{b}{a}.$

Whence $x\,(= \tfrac{1}{2}a - \tfrac{1}{2}z) = \tfrac{1}{4}a + \tfrac{1}{4}\,\dfrac{b}{a}.$

And $y\,(= \sqrt{x^2 - z^2}) = \tfrac{1}{4}\sqrt{10b - 3a^2 - \dfrac{3b^2}{a^2}}.$

PROPOSITION XIV.

25. All compound equations may be reduced by bringing the ſame unknown quantities, in different equations, to have the

ſame coefficients by multiplication or diviſion; and then adding and ſubtracting theſe equations. This reduction may alſo ſometimes take place, by only multiplying or dividing ſome of the equations by a proper quantity, (Ax. VII, VIII, IX and X.)

Ex. 1. Suppoſe $\begin{cases} x+y=s \\ x-y=d \end{cases}$

Then by Addition, $2x=s+d$, and $x=\frac{s+d}{2}$

Alſo by Subtraction, $2y=s-d$, and $y=\frac{s-d}{2}$

Ex. 2. Suppoſe $\begin{cases} x+y+z=26 \\ x-y \quad\ =4 \\ x-z \quad\ =6 \end{cases}$

Then by Addition, $3x=36$, and $x=12$.
Whence $y=8$, and $z=6$.

Ex. 3. Suppoſe $\begin{cases} ax=\dfrac{b^2x-b^3}{z} \\ x=\dfrac{az}{x-b} \end{cases}$

Then by Multiplication, $ax^2=ab^2$, or $x=b$.

Ex. 4. Suppoſe $\begin{Bmatrix} x+y=12 \\ 5x+3y=50 \end{Bmatrix}$ Then $\begin{cases} 5x+5y=60 \\ 5x+3y=50 \end{cases}$

Whence by Subtraction, $2y=10$; thus: $y=5$, and $x=7$.

Ex. 5.

Suppoſe $\begin{Bmatrix} 5x+8y=124 \\ 3x-2y=20 \end{Bmatrix}$ Then $\begin{Bmatrix} 15x+24y=372 \\ 15x-10y=100 \end{Bmatrix}$ And $\begin{cases} 10x+16y=248 \\ 24x-16y=160 \end{cases}$

Whence by Subtraction, $34y=272$; and by Addition, $34x=408$: Thus: $y=8$, and $x=12$.

Ex. 6. Suppoſe $\begin{cases} 5x-3y=90 \\ 2x+5y=160. \end{cases}$

Ex. 7. Suppoſe $\left\{\begin{array}{l}\frac{x}{2}-12=\frac{y}{4}+8\\ \frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27.\end{array}\right.$

Ex. 8. Suppoſe $\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62\\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47\\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38.\end{array}\right.$

PROPOSITION XV.

26. Affected quadratic equations, (18) may be reduced, by adding, in the firſt and ſecond caſes, and ſubtracting in the third caſe, both ſides of the equations to and from the ſquare of half the coefficient of the term, involving the variable quantity itſelf, (12); and then extracting the ſquare root on both ſides. (Ax. XI.)

Caſe I. $x^2+px=rq\ \ldots\ x=-\frac{1}{2}p+\sqrt{\frac{1}{4}p^2+rq}=\frac{-p+\sqrt{p^2+4rq}}{2}$

Caſe II. $x^2-px=rq\ \ldots\ x=\frac{1}{2}p+\sqrt{\frac{1}{4}p^2+rq}=\frac{p+\sqrt{p^2+4rq}}{2}.$

Caſe III. $px-x^2=rq\ \ldots\ x=\frac{1}{2}p\pm\sqrt{\frac{1}{4}p^2-rq}=\frac{p\pm\sqrt{p^2-4rq}}{2}.$

Ex. 1. $x^2+6x=40$ . . . . . . $x=4.$

Ex. 2. $x^2-x=56$ . . . . . . $x=8$

Ex. 3. $20x-x^2=96$ . . . . . $x=12$, and 8.

Ex. 4. $\frac{5}{2}x-x^2=\frac{3}{2}$ . . . . . . $x=1\frac{1}{2}$, and 1.

Ex. 5. Suppoſe $x^2+\frac{5}{2}x=\frac{33}{2}.$

Ex. 6. Suppoſe $x^2-\frac{1}{2}x=22\frac{1}{2}.$

Ex. 7. Suppoſe $x^2+\frac{1}{2}x=39.$

Ex. 8. Suppoſe $x^2+10x=20.$

Ex. 9. Suppoſe $x^2+6x=11.$

Ex. 10. Suppoſe $x^2+\frac{1}{12}x=\frac{107}{96}.$

Scholium I.

27. The third caſe of affected quadratic equations, is ſaid to be **Ambiguous**, becauſe the root (x), admits of *two different interpretations*, viz. one leſs than $\frac{1}{2}p$, and the other greater than $\frac{1}{2}p$, but leſs than p. From the equation, $px - x^2 = rq$, it is evident, that x is leſs than p; ſuppoſe therefore $x = \frac{1}{2}p$, then by ſubſtitution, $\frac{1}{4}p^2 = rq$, and in every other caſe $\frac{1}{4}p^2$, is *greater* than rq: Hence, then in completing the ſquare, both ſides of the equation muſt be ſubtracted from $\frac{1}{4}p^2$, which gives $\frac{1}{4}p^2 - px + x^2 = \frac{1}{4}p^2 - rq$, and therefore $\sqrt{\frac{1}{4}p^2 - px + x^2} = \sqrt{\frac{1}{4}p^2 - rq}$, (Ax. XI.); but the ſquare root of $\frac{1}{4}p^2 - px + x^2$, may be $\frac{1}{2}p - x$, or, $x - \frac{1}{2}p$ (12), and thus: $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - rq}$, according to the propoſition.

28. Since the *greateſt* magnitude of the quantity $\overline{p - x} \times x$ ($= px - x^2$) is $\frac{1}{4}p^2$, which ariſes by ſuppoſing $p - x = x$, or, $x = \frac{1}{2}p$: Therefore, as $\overline{p - x} \times x$ may repreſent any product whoſe increaſe is *limited*, the factors, $p - x$, and x, being parts of the ſame quantity p, it appears, that the greateſt magnitude of any ſuch product, may be determined, by *ſuppoſing the factors equal, and ſubſtituting in the product the value of the variable quantity derived from this equation.*

Scholium II.

29. Since the exponent of the ſquare is always double that of the root (9), any equations falling under the following general forms, *viz.* $x^{2m} \pm px^m = rq$, and $px^m - x^{2m} = rq$, may be reduced to quadratics by ſubſtitution. Thus: Let $y = x^m$; then $y^2 = x^{2m}$ (9); and by Caſe I. and II. (26), in the equation $x^{2m} \pm px^m = rq$, we have

$$x = \sqrt[m]{\pm\tfrac{1}{2}p + \sqrt{\tfrac{1}{4}p^2 + rq}} = \sqrt[m]{\frac{\pm p + \sqrt{p^2 + 4rq}}{2}}.$$

Alſo, in the equation $px_m - x^2{}_m = rq$, by Caſe III. (26), we have

$$x = \sqrt[m]{\tfrac{1}{2}p \pm \sqrt{\tfrac{1}{4}p^2 - rq}} = \sqrt[m]{\frac{p \pm \sqrt{p^2 - 4rq}}{2}}.$$

Ex. 1. Suppoſe $27x^2 - x^4 = 176$; then $x = 4$, and $x = \sqrt{11}$.

Ex. 2. Suppoſe $x^6 + 4x^3 = 96$; then $x = 2$.

Ex. 3. Suppo $x^8 - 10x^4 = 5751$.

Ex. 4. Suppoſe $x^3 - 14x^{\frac{3}{2}} = 16$.

30. In reſolving the equations $x^{2m} \pm px^m = rq$, and $px^m - x^{2m} = rq$, by reducing them to quadratics, the following rule, will, in many caſes, be uſeful, [redacted] Let $R \pm \sqrt{S}$ repreſent any quadratic binomial, involving a radical quantity, *viz.* $\sqrt{S}$; ſuppoſe

$\sqrt{R^2 - S} = Q$, and $\dfrac{Q + R}{2} = B^2$, then will $B \pm \sqrt{B^2 - Q}$ or

$\sqrt{B^2 - Q} \pm B$ be the ſquare root of the propoſed binomial $R \pm \sqrt{S}$.

Ex. 1. Required the ſquare root of $27 \pm \sqrt{704}$? Here $Q = 5$, $B = 4$, and therefore $4 \pm \sqrt{11}$ the root required.

Ex. 2. Required the ſquare root of $-1 \pm \sqrt{-8}$? Here $Q = 3$, $B = 1$, and therefore $1 \pm \sqrt{-2}$ the root required.

Ex. 3. Required the ſquare root of $3 \pm \sqrt{8}$?

Ex. 4. Required the ſquare root of $6 \pm \sqrt{20}$?

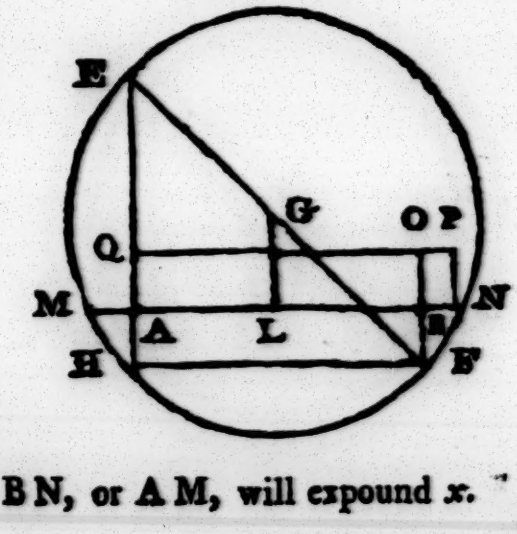
PROPOSITION XVI.

31. The root or roots of any affected quadratic equation, may be determined by means of a circle.

Caſe I. $x^2 + px = rq$.

CONSTRUCTION.

Draw the right line A B equal to p, and at right angles thereto, but on contrary ſides, the two lines A E, B F, the former being equal to r, and the latter equal to q; join the points E F, and biſect E F in G; from the center G, with the radius G E, deſcribe a circle meeting A E again in H, and cutting A B produced in M, N: Thus B N, or A M, will expound x.



Caſe II. $x^2 - px = rq$.

CONSTRUCTION.

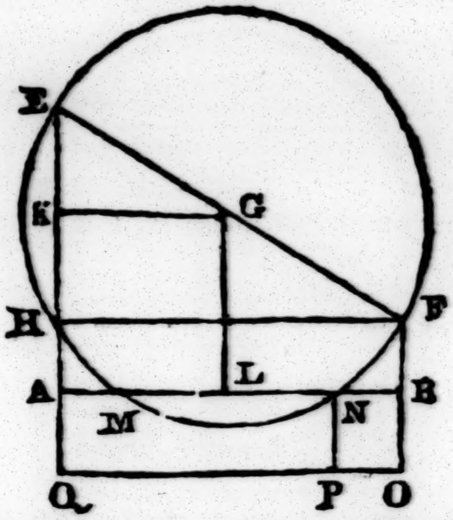
This caſe is conſtructed as the former; the value of x being MB, or AN.

Caſe III. $px - x^2 = rq$.

DETERMINATION.

In this caſe rq is not greater than $\frac{1}{4}p^2$ (27.)

CONSTRUCTION.



Draw the right line AB equal to p, and perpendicular thereto, and on the ſame ſide the two lines AE, BF, the former being equal to r, and the latter equal to q; join the points E, F, and biſect EF in G; from the center G with radius GE, deſcribe a circle cutting AE again in H; join the points H, F, and draw GK, parallel to HF, as alſo GL parallel to AE, and meeting AB in L: Then, if GL be equal to GE, the right line, AB, will touch the circle in L, (16. e. 3.), and AL, or LB, will expound x: But if GL be not equal to GE, it muſt be leſs by the *determination*, (and 6. e. 2.), and of conſequence the circle EHF, will interſect the line AB in two points, as M, N; and the lines AN, AM, will be the values of x*.

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* Theſe conſtructions were firſt given by Willebrordus Snellius, and the reader may conſult Dr Simſon's Notes upon book VI. of Euclid's Elements. And for a full illuſtration of all the propoſitions hitherto given for the reduction of equations, the reader may have recourſe to Sir Iſaac Newton's Arithmetica Univerſalis, Maclaurin's Algebra, Saunderſon's Algebra, or the Abridgement of the ſame, Simpſon's Algebra, Simpſon's Select Exerciſes, [illegible] s' Graveſande's Algebra, &c.

PROPOSITION XVII.

32. In the general affected cubic equation, $z^3 - 3qz = 2r$, or $z^3 - 3qz - 2r = 0$, where the term involving the indeterminate quantity, z^2 is wanting; the value of z is *greater* than $\sqrt[3]{2r}$, as alſo *greater* than $\sqrt{3q}$, and may be expreſſed univerſally in manner following, *viz.*

$$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$$

Or,

$$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \frac{q}{\sqrt[3]{r + \sqrt{r^2 - q^3}}}.$$

DEMONSTRATION.

Aſſume two quantities y, v, whereof y is the greater; and ſuppoſe theſe quantities to be ſuch, that $y + v = z$, and $yv = q$: Then, by ſubſtituting in the propoſed cubic equation, we have $2ry^3 - y^6 = q^3$, which gives $y^3 = r \pm \sqrt{r^2 - q^3}$ (29, 27), and of conſequence $v^3 = r \mp \sqrt{r^2 - q^3}$; but ſince by ſuppoſition, y is greater than v, therefore $y^3 = r + \sqrt{r^2 - q^3}$, and $v^3 = r - \sqrt{r^2 - q^3}$, and hence

$$(y + v =)\, z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$$

Or,

$$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \frac{q}{\sqrt[3]{r + \sqrt{r^2 - q^3}}}.$$

Q. E. D.

33. Corol. 1. From the expreſſion for z, it appears, that if r^2 ſhould be leſs than q^3, the quantity $r^2 - q^3$, would be negative, (4) and its ſquare root, therefore imaginary, (5): So that in this caſe the value of z cannot be directly determined by the general method of evolution, (11).

34. Corol. 2. If we change the ſign of the term involving z, the equation becomes $z^3 + 3qz - 2r = 0$; where z is *leſs* than $\sqrt[3]{2r}$, as alſo *leſs* than $\sqrt{3q}$: Here then, the quantity $3q$ having a different ſign from that in the equation of this propoſition,

tion, therefore in the general expreſſion for z, the ſign of q, and of conſequence that of q^3, (5), muſt be changed; and hence in this caſe,

$$z = \sqrt[3]{r + \sqrt{r^2 + q^3}} - \sqrt[3]{r - \sqrt{r^2 + q^3}}.$$

Or,

$$z = \sqrt[3]{r + \sqrt{r^2 + q^3}} - \frac{q}{\sqrt[3]{r + \sqrt{r^2 + q^3}}}.$$

35. Corol. 3. If we change the ſign of $(2r)$, the abſolute term, the equation becomes $3qz - z^3 = 2r$, or $z^3 - 3qz + 2r = 0$; where if z be not equal to $\sqrt{q}$, it will have TWO REAL VALUES, one *leſs* than $\sqrt{q}$, and the other *greater* than $\sqrt{q}$, but *leſs* than $\sqrt{3q}$: Comparing this equation with that reſolved in this propoſition, it appears, that in the general expreſſion for z, (32), the ſign of the quantity r, muſt be changed, and hence in this caſe,

$$z = \sqrt[3]{-r + \sqrt{r^2 - q^3}} + \sqrt[3]{-r - \sqrt{r^2 - q^3}}.$$

Or,

$$z = \sqrt[3]{-r + \sqrt{r^2 - q^3}} + \frac{q}{\sqrt[3]{-r + \sqrt{r^2 - q^3}}}.$$

But in this equation, ſince the quantity r^2 is *leſs* than the quantity q^3, it is evident, that the value of z cannot be directly determined by the general method of evolution, (11).

The general reſolution of this equation, and of that in the firſt corollary, (33), will be given hereafter.

36. Corol. 4. By ſubſtituting $x - p$, inſtead of z in the equation, (32), $z^3 - 3qz - 2r = 0$, there will be produced the following cubic equation complete in all its terms, *viz.*

$$x^3 - 3px^2 + \overline{3p^2 - 3q} \times x - \overline{2r + p^3 - 3pq} = 0,$$

And from this general equation, the ſolution of any complete cubic equation, may be derived, by comparing the coefficients, and taking proper values of p, q, and r.

EXAMPLES OF CUBIC EQUATIONS.

(32.) *Ex.* 1. Suppoſe $z^3 - 36z = 91$. Here $q = 12$, $r = \frac{91}{2}$, and therefore $z = 4 + 3 = 7$.

Ex. 2. Suppoſe $z^3 - 12\,z = 16$. Here $q = 4$, $r = 8$, and $r^2 - q^3 = 0$; whence $z = 2\sqrt[3]{r} = 4$.

Ex. 3. Suppoſe $z^3 - 96\,z = 576$.

Ex. 4. Suppoſe $z^3 - 2\,z = 21$.

(34.) *Ex.* 5. Suppoſe $z^3 + 6\,z = 20$. Here $q = 2$, $r = 10$, and therefore $z = 2$.

Ex. 6. Suppoſe $z^3 + 30\,z = 117$. Here $q = 10$, $r = \frac{117}{2}$, and therefore $z = 5 - 2 = 3$.

Ex. 7 Suppoſe $z^3 + 9\,z = 30$.

Ex. 8. Suppoſe $z^3 + 24\,z = 58,7914$.

(36.) *Ex.* 9. Suppoſe $x^3 - 17\,x^2 + 54\,x - 350 = 0$. Here $p = \frac{17}{3}$, $3q = 42$, $2r = 407.92$, and the equation for z, is $z^3 - 42\,z = 407.92$, where $z = 9.287401$ nearly (32), and therefore $x\ (= z + p) = 14.954067$ nearly.

Ex. 10. Suppoſe $x^3 + 6\,x^2 - 183\,x - 2704 = 0$. Here $(-p = 2$, or$)$, $p = -2$, $3q = 105$, $2r = 2322$, and the equation for z, is $z^3 - 195\,z = 2322$, where $z = 18$, and therefore $x\ (= z + p) = 16$.

Ex. 11. Suppoſe $x^3 + 15\,x^2 + 84\,x - 100 = 0$. Here $p = -5$, $3q = -9$, $2r = 270$, and the equation for z, is $z^3 + 9\,z = 270$, where (34), $z = 6$, and therefore $x\ (= z + p) = 1$.

Ex. 12. Suppoſe $x^3 - 12\,x^2 + 40\,x - 512 = 0$.

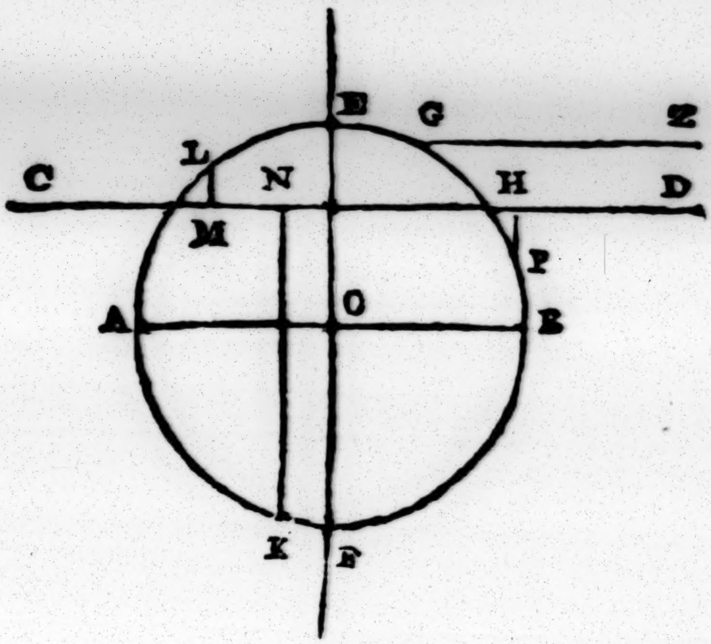
Ex. 13. Suppoſe $x^3 + 74\,x^2 + 8729\,x - 560783 = 0$.

PROPOSITION XVIII.

37. *Admitting* affirmative and negative quantities, the three roots belonging to thoſe caſes of affected cubic equations, which could

not be reſolved by the laſt propoſition, may be determined by means of a circle from the following

CONSTRUCTION.



General Equation. $\begin{cases} x^3 \pm 3px^2 \pm \overline{3p^2-3q} \times x + \overline{3pq-p^3-2r} = 0 \\ x^3 \pm 3px^2 \pm \overline{3p^2-3q} \times x - \overline{2r} + \overline{p^3}\,\overline{-3pq} = 0 \end{cases}$

With a radius A O equal to $2\sqrt{q}$, deſcribe the circle A E B F, and draw the perpendicular diameters A B, E F; parallel to A B, and at the diſtance of p therefrom, draw C D above or below A B, according as the ſign of the *ſecond* term of equation is affirmative or negative; alſo parallel to A B, and at the diſtance of $\frac{2r}{q}$, therefrom draw Z G above or below A B, according as the ſign of the *abſolute* term is affirmative or negative; take the arch B P equal to one third of the arch B G, and from the point P divide the periphery of the circle into three equal parts P L, L K, and K P; from the points of diviſion P, L, K, draw to C D the perpendicular lines P H, L M, K N, and theſe will be the three roots of the equation; the affirmative roots falling above, and the negative roots below C D.

38. Corol. 1. If $p = 0$, then C D coincides with A B; and if $r = 0$, then Z G coincides with A B.

39. Corol 2. When $\frac{2r}{q}$ is greater than $2\sqrt{q}$, the conſtruction fails.

Ex. 1. $x^3 - 12x^2 + 41x - 42 = 0$.
Here $p = 4$, $q = \frac{7}{3}$, and $r = 3$; whence $2\sqrt{q} = 3.054 = 3$ nearly, $\frac{2r}{q} = 2.571 = 2\frac{1}{2}$ nearly; and therefore $LM = 2$, $PH = 3$, and $NK = 7$.

Ex. 2. $x^3 + 9x^2 - 22x - 120 = 0$.

Here $p = 3$, $q = \frac{49}{3}$, and $r = 0$; whence $2\sqrt{q} = \frac{14}{\sqrt{3}} = 8.1$ nearly; and ſince, in this example, Z G coincides with A B (38), therefore $LM = 4$, $BH = -3$, and $MK = -10$.

Ex. 3. $x^3 - 17x^2 + 82x - 120 = 0$.

Here $p = \frac{17}{3} = 5\frac{2}{3}$, $q = \frac{43}{9}$, $2\sqrt{q} = 4.4$ nearly, and $\frac{2r}{q} = 4$ nearly; whence $KN = 3$, $PH = 4$, and $LM = 10$.

Ex. 4. $x^3 - 13x + 12 = 0$.

Here $p = 0$, $q = \frac{13}{3}$, $2\sqrt{q} = 4.2$ nearly, and $\frac{2r}{q} = 2.8$ nearly; thus, in this example, C D coincides with A B, and $PH = 1$, $LM = 3$, and $NK = -4$.

Ex. 5. $x^3 - \frac{7}{4}x + \frac{3}{4} = 0$.

Here $p = 0$, $q = \frac{7}{12}$, $2\sqrt{q} = 1.6$ nearly, and $\frac{2r}{q} = \frac{9}{7} = 1.3$ nearly; thus, in this example, C D coincides with A B, and $PH = \frac{1}{2}$, $LM = 1$, and $NK = -1\frac{1}{2}$.

PPOPOSITION XIX.

40. If one ſide of an equation is a fraction involving CONSTANT quantities *only*, and the other ſide a fraction involving ONE VARIABLE quantity in *each* of its terms, the *ſeveral* values of theſe variable quantities will be determined by taking all poſſible equimultiples of the correſpondent terms of the given fraction.

Let $m\,x = n\,y$ repreſent any equation involving two variable quantities; then, ſince $\frac{x}{y} = \frac{n}{m}$ (Ax. X.) $= \frac{2\,n}{2\,m} = \frac{3\,n}{3\,m} = \frac{4\,n}{4\,m}$, &c. $= \frac{\frac{1}{2}n}{\frac{1}{2}m} = \frac{\frac{1}{3}n}{\frac{1}{3}m} = \frac{\frac{1}{4}n}{\frac{1}{4}m}$, &c. (6), it is obvious, that, if

$x = n,\ 2\,n,\ 3\,n,\ 4\,n,$ &c. $\frac{1}{2}n,\ \frac{1}{3}n,\ \frac{1}{4}n,$ &c.

then $y = m,\ 2\,m,\ 3\,m,\ 4\,m,$ &c. $\frac{1}{2}m,\ \frac{1}{3}m,\ \frac{1}{4}m,$ &c.

(Ax. X.) Q. E. D.

41. Corol. 1. When an equation involves two variable quantities, it is ſaid to be INDETERMINATE; becauſe each of the variable quantities may be interpreted by ſeveral different numbers, and ſometimes by the ſame numbers.

42. Corol. 2. From this propoſition, it appears, that in the equation $m\,x = n\,y$, if x can be interpreted by n or any multiple of n, then may y be interpreted by m or an equimultiple of m. That is, in an indeterminate equation, involving two variable quantities, x, y, the values of x go on by n, the coefficient of y, and thoſe of y by m, the coefficient of x: And therefore, having obtained the firſt values of x, y, the other values will be determined by a conſtant addition or ſubtraction of the coefficients of y, x, or of equimultiples of theſe quantities.

43. Corol. 3. Let A, B, C, repreſent three conſtant quantities; then, if $A\,x + B\,y = C$, it is evident, that, if x increaſes, y muſt decreaſe, and *vice verſa*: But, if $A\,x - B\,y = C$, the two variable quantities, x, y, muſt either, both together, encreaſe, or both together decreaſe.

44. Corol. 4. From the equation $A\,x \pm B\,y = C$, we have $x = \frac{C \mp B\,y}{A}$ and $y = \frac{A\,x \backsim C}{B}$, (Ax. X.) And therefore, if by actual diviſion, either of theſe quantities be reduced to more ſimple terms, then, by ſubſtituting for the fractional part of the quotient, a new equation will be obtained more ſimple than the original one.

EXAMPLES OF INDETERMINATE EQUATIONS.

Ex. 1. Suppoſe $9\,x - 7\,y = 6$.

Here then we have $\frac{3\,x - 2}{y} = \frac{7}{3}$, and therefore, by this propoſition (40.)

If $3x - 2 = 7,\ 14,\ 21,\ 28,$ &c.

Then $y = 3,\ 6,\ 9,\ 12,$ &c.

Whence, ſuppoſing x, y, to be whole numbers

$$\left.\begin{array}{r} 3x - 2 = 7 \\ y = 3 \end{array}\right\}\begin{array}{l} x = 3 \\ y = 3. \end{array}$$

Thus Corol. 2. (42.) and Corol. 3. (43.)

$x = 3,\ 10,\ 17,\ 24,\ 31,\ 38,$ &c. (Difference 7.)

$y = 3,\ 12,\ 21,\ 30,\ 39,\ 48,$ &c. (Do. 9.)

Ex. 2. Suppoſe $5x + 9y = 200$.

Here we have $\dfrac{40 - x}{y} = \dfrac{9}{5}$, and therefore,

If $40 - x = 9,\ 18,\ 27,\ 36,$

Then $y = 5,\ 10,\ 15,\ 20.$

Whence, ſince the firſt ſeries terminates at 36, the quantities x, y, are limited to four values, in whole Numbers.

$$\left.\begin{array}{r} 40 - x = 9 \\ y = 5 \end{array}\right\}\begin{array}{l} x = 31 \\ y = 5. \end{array}$$

Thus: $x = 31,\ 22,\ 13,\ 4,$ (Difference 9.)

$y = 5,\ 10,\ 15,\ 20,$ (Do. 5.)

Ex. 3. Suppoſe $2x - y = 100$.

Here we have $\dfrac{x}{100 + y} = \frac{1}{2}$, and therefore,

If $x = 1$ . . . . $51,\ 52,\ 53,$ &c.

Then $100 + y = 2$ . . . . . $102,\ 104,\ 106,$ &c.

Whence, ſuppoſing $\left.\begin{array}{r} x = 51 \\ 100 + y = 102 \end{array}\right\}$ Then $\left\{\begin{array}{l} x = 51 \\ y = 2. \end{array}\right.$

Thus: $x = 51,\ 52,\ 53,\ 54,\ 55,$ &c. (Diff. 1.)

$y = 2,\ 4,\ 6,\ 8,\ 10,$ &c. (Do. 2.)

Or, by Corol. 4. (44.)

Since $2x - y = 100$, therefore $x = 50 + \frac{1}{2}y$; let $v = \frac{1}{2}y$, or $2v = y$; then, ſince $\frac{v}{y} = \frac{1}{2}$, therefore,

$v = 1,\ 2,\ 3,\ 4,\ 5,\ 6,$ &c. (Diff. 1)
$y = 2,\ 4,\ 6,\ 8,\ 10,\ 12,$ &c. (Do. 2.)
$(50 + \frac{1}{2}y =)\, x = 51,\ 52,\ 53,\ 54,\ 55,\ 56,$ &c. (Do. 1.)

Ex. 4. Suppoſe $14x - 19y = 11$.

Here, we have $\frac{14x - 11}{y} = \frac{19}{1}$, and therefore,

If $14x - 11 = 19,\ 38,\ 57,\ 76,\ 95,\ 114,\ 133,\ 152,\ 171,$ &c.
Then $y = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,$ &c.

Whence, ſuppoſing x, y, to be whole numbers,

$$\left.\begin{array}{r}14x - 11 = 171\\ y = 9\end{array}\right\}\ \text{Then}\ \left\{\begin{array}{l}x = 13\\ y = 9.\end{array}\right.$$

Thus: $x = 13,\ 32,\ 51,\ 70.\ 89,$ &c. (Diff. 19.)
$y = 9,\ 23,\ 37,\ 51,\ 65,$ &c. (Do. 14.)

Or, by Corol. 4. (44.)

Since $14x = 19y + 11$, therefore, $x = \frac{19y + 11}{14} = y + \frac{5y + 11}{14}$; let $v = \frac{5y + 11}{14}$, then $y = \frac{14v - 11}{5} = 2 \times \overline{v - 1} + \frac{4v - 1}{5}$; let $z = \frac{4v - 1}{5}$, then $v = \frac{5z + 1}{4} = z + \frac{z + 1}{4}$; let $w = \frac{z + 1}{4}$, then $\frac{z + 1}{w} = \frac{4}{1}$, or $\frac{w}{z + 1} = \frac{1}{4}$; and therefore,

$$\text{If}\ \left.\begin{array}{r}w = 1\\ z + 1 = 4\end{array}\right\}\ \text{Then}\ \left\{\begin{array}{l}w = 1\\ z = 3\end{array}\right\}\ \text{And}\ v\ (= z + w) = 4.$$

Whence, $y\ (= 2 \times \overline{v - 1} + z) = 9$, and $x\ (= y + v) = 13$. And thus, the firſt values of x, y, may be brought to the moſt *ſimple determination poſſible.*

Ex. 5. Suppoſe $21x + 5y = 800$.

Ex. 6. Suppoſe $8y - 3x = 16$.

Ex. 7. Suppoſe $24x - 13y = 16$.

45. Corol. 5. By this propoſition, we can determine the leaſt whole numbers, that, divided by given numbers, ſhall have given remainders. Thus. Suppoſe we were to find the leaſt whole number, that, divided by 3, 7, there ſhall remain 1, 3; let N repreſent the number ſought, then $\frac{N-1}{3}$, $\frac{N-3}{7}$, are whole numbers; wherefore, ſuppoſe $\frac{N-1}{3} = P$, then $N = 3P + 1$, and $\frac{N-3}{7} = \frac{3P-2}{7}$; ſuppoſe $\frac{3P-2}{7} = Q$, then $P = \frac{7Q+2}{3} = 2Q + \frac{Q+2}{3}$; let $\frac{Q+2}{3} = R$, then $\frac{Q+2}{R} = \frac{3}{1}$, and therefore,

$$\left.\begin{array}{r} Q+2=3 \\ R=1 \end{array}\right\} \text{ Then } \left\{\begin{array}{l} Q=1 \\ R=1 \end{array}\right., \text{ and therefore } P=3.$$

Whence, $N = 10$ the leaſt whole number required. And by continuing the ſeries for $Q + 2$, R, the other values of N may be obtained.

Or,

Since $P = 2Q + \frac{Q+2}{3}$; let $\frac{Q+2}{3} = 1$, or $Q = 1$, then $P = 3$, and $N = 10$, as before. And, by ſuppoſing $\frac{Q+2}{3} = 2, 3, 4$, &c. the next greater values of N will be determined as before.

Again, Suppoſe we were to find the leaſt whole number, that, divided by 28, 19, ſhall give 10, 12, for remainders. Let N repreſent the number required, then $\frac{N-10}{28}$, $\frac{N-12}{19}$, are whole numbers by ſuppoſition; wherefore ſuppoſe $\frac{N-10}{28} = P$, then $N = 28P + 10$, and therefore $\frac{N-12}{19} = \frac{28P-2}{19} = P + \frac{9P-2}{19}$; let $\frac{9P-2}{19} = Q$, then $P = \frac{19Q+2}{9} = 2Q + \frac{Q+2}{9}$: Here, by taking $\frac{Q+2}{9} = 1$, we have $Q = 7$, and $N = 430$, the leaſt number required; and by taking $\frac{Q+2}{9} = 2, 3$, *&c.* the next greater values of N will be obtained; or, the ſame may be determined, by forming the progreſſions, as in the laſt example.

46. The analyſis given in this Corollary, may be extended to *any* number of diviſors and remainders.

Ex. 1. To find a whole number, which, being divided by 3, 5, 7, 2, there ſhall remain 2, 4, 6, 0, reſpectively.

Let N repreſent the number required; then by ſuppoſition, $\frac{N-2}{3}$, $\frac{N-4}{5}$, $\frac{N-6}{7}$, and $\frac{N-0}{2}$ are whole numbers. Wherefore, ſuppoſe $\frac{N-2}{3} = P$; then $N = 3P + 2$, and therefore,

$$\frac{N-4}{5} = \frac{3P-2}{5} = \frac{5P - \overline{2P+2}}{5} = P - 2 \times \frac{P+1}{5}.$$

Suppoſe now $\frac{P+1}{5} = Q$; then $N = 15Q - 1$, and therefore $\frac{N-6}{7} = \frac{15Q-7}{7} = 2Q - 1 + \frac{R}{7}$.

Suppoſe $\frac{R}{7} = S$; then $N = 105S - 1$, and therefore $\frac{N-0}{2} = 52S + \frac{S-1}{2}$.

Let $\frac{S-1}{2} = 0$; then $S = 1$, and $N = 104$, the *leaſt* whole number required: And, by ſuppoſing $\frac{S-1}{2}=1, 2, 3,$ *&c.* the next ſucceeding values of N will be determined.

Ex. 2. To find a *whole* number, that, being divided by 16, 17, 18, 19, 20, there ſhall remain 6, 7, 8, 9, 10, reſpectively.

Let N repreſent the number required; then by ſuppoſition, $\frac{N-6}{16}$, $\frac{N-7}{17}$, $\frac{N-8}{18}$, $\frac{N-9}{19}$, and $\frac{N-10}{20}$, are whole numbers.

Suppoſe $\frac{N-6}{16}=P$; then $N=16P+6$, and therefore,

$$\frac{N-7}{17}=\frac{16P-1}{17}=\frac{17P-\overline{P+1}}{17}=P-\frac{P+1}{17}.$$

Suppoſe now $\frac{P+1}{17}=Q$; then $N=272Q-10$, and therefore $\frac{N-8}{18}=\frac{272Q-18}{18}=15Q-1+\frac{Q}{9}.$

Suppoſe now $\frac{Q}{9}=R$; then $N=2448R-10$, and therefore $\frac{N-9}{19}=\frac{2448R-19}{19}=128R-1+16\times\frac{R}{19}.$

Suppoſe $\frac{R}{19}=S$; then $N=46512S-10$, and therefore,

$$\frac{N-10}{20}=\frac{46512S-20}{20}=2325S-1+4\times\frac{S}{5}.$$

Let $\frac{S}{5}=1$; then $N=232550$, the *leaſt* whole number required: And, by ſuppoſing $\frac{S}{5}=2, 3, 4,$ *&c.* the next ſucceeding values of N will be obtained *.

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* See Dodſon's Repoſitory, v. 1. Simpſon's Algebra and Select Exerciſes; as alſo Emerſon's Algebra.

47. Corol. 6. Any equation, involving two diffeaent powers of the ſame variable quantity, may be reduced by ſubſtitution to the form of an indeterminate equation, involving two variable quantities: And, *hence the reſolution of all commenſurate quadratic equations, as alſo of all commenſurate cubic equations, wanting one term, biquadratic equations wanting two terms*, &c.

EXAMPLES OF QUADRATIC EQUATIONS, (26, 27)

Table of Square Numbers.

Root	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12.
Square	1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	121,	144.

Ex. 1. $x^2 + 6x = 40$. Here $\frac{40 - x^2}{x} = \frac{6}{1}$

$$\left.\begin{array}{r} 40 - x^2 = 6,\ 12,\ 18,\ 24 \\ x = 1,\ 2,\ 3,\ 4 \end{array}\right\} \begin{array}{r} 40 - 24 = 16 \\ x = 4. \end{array}$$

Ex. 2. $x^2 + 3x = 88$. Here $\frac{88 - x^2}{x} = \frac{3}{1}$.

$$\left.\begin{array}{r} 88 - x^2 = 3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ 24 \\ x = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \end{array}\right\} \begin{array}{r} 88 - 24 = 64 \\ x = 8. \end{array}$$

Ex. 3. $x^2 - \frac{1}{2}x = 22\frac{1}{2}$. Here $\frac{x^2 - 22\frac{1}{2}}{x} = \frac{\frac{1}{2}}{1}$

$$\left.\begin{array}{r} x^2 - 22\frac{1}{2} = \frac{1}{2},\ 1,\ 1\frac{1}{2},\ 2,\ 2\frac{1}{2} \\ x = 1,\ 2,\ 3,\ 4,\ 5 \end{array}\right\} \begin{array}{r} 2\frac{1}{2} + 22\frac{1}{2} = 25 \\ x = 5. \end{array}$$

Ex. 4. $5x - x^2 = 6$. Here $\frac{x^2 + 6}{x} = \frac{5}{1}$

$$\left.\begin{array}{r} x^2 + 6 = 5,\ 10 \\ x = 1,\ 2 \end{array}\right\} \begin{array}{r} 10 - 6 = 4 \\ x = 2 \end{array} \left\{\begin{array}{c} 15 \\ 3 \end{array}\right\} \begin{array}{r} 15 - 6 = 9 \\ x = 3. \end{array}$$

Ex. 5. $x^6 + 4x^3 = 96$. Let $v = x^3$, then $v^2 + 4v = 96$ (27), and $\frac{96 - v^2}{v} = \frac{4}{1}$

$$\left.\begin{array}{r} 96 - v^2 = 4,\ 8,\ 12,\ 16,\ 20,\ 24,\ 28,\ 32 \\ v = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \end{array}\right\} \begin{array}{r} 96 - 32 = 64 \\ v = 8. \end{array}$$

And $x = \sqrt[3]{v} = 2$.

Or,

Since $v^2 + 4v = 96$, therefore $\frac{v^2}{24 - v} = \frac{4}{1}$

$$\left.\begin{array}{r} v^2 = 4,\ 16,\ 36,\ 64 \\ 24 - v = 1,\ 4,\ 9,\ 16 \end{array}\right\} v = 8 \text{ as before.}$$

EXAMPLES OF DEFICIENT CUBIC EQUATIONS.

(32, 33, 34, 35.)

Table of Cube Numbers.

Root	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12.
Cube	1,	8,	27,	64,	125,	216,	343,	512,	729,	1000,	1331,	1728.

Ex. 1. $z^3 - 6z = 40$. Here $\frac{z^3 - 40}{z} = \frac{6}{1}$

$$\left.\begin{array}{r} z^3 - 40 = 6,\ 12,\ 18,\ 24 \\ z = 1,\ 2,\ 3,\ 4 \end{array}\right\} \begin{array}{r} 24 + 40 = 64 \\ z = 4. \end{array}$$

Ex. 2. $z^3 + 84z = 400$. Here $\frac{400 - z^3}{z} = \frac{84}{1}$

$$\left.\begin{array}{r} 400 - z^3 = 84,\ 168,\ 252,\ 336 \\ z = 1,\ 2,\ 3,\ 4 \end{array}\right\} \begin{array}{r} 400 - 336 = 64 \\ z = 4. \end{array}$$

Ex. 3. $z^3 + 12z = 32$. Here $\dfrac{32 - z^3}{z} = \dfrac{12}{1}$

$$\left.\begin{array}{r} 32 - z^3 = 12,\ 24 \\ z = 1,\ \ 2 \end{array}\right\} \begin{array}{r} 32 - 24 = 8 \\ z = 2. \end{array}$$

(33.) *Ex.* 4. $z^3 - 7z = 6$. Here $\dfrac{z^3 - 6}{z} = \dfrac{7}{1}$

$$\left.\begin{array}{r} z^3 - 6 = 7,\ 14,\ 21 \\ z = 1,\ \ 2,\ \ 3 \end{array}\right\} \begin{array}{r} 21 + 6 = 27 \\ z = 3. \end{array}$$

(33.) *Ex.* 5. $z^3 - 13z = 12$. Here $\dfrac{z^3 - 12}{z} = \dfrac{13}{1}$

$$\left.\begin{array}{r} z^3 - 12 = 13,\ 26,\ 39,\ 52 \\ z = 1,\ \ 2,\ \ 3,\ \ 4 \end{array}\right\} \begin{array}{r} 52 + 12 = 64 \\ z = 4. \end{array}$$

(35.) *Ex.* 6. $21z - z^3 = 20$. Here $\dfrac{z^3 + 20}{z} = \dfrac{21}{1}$

$$\left.\begin{array}{r} z^3 + 20 = 21 \\ z = 1 \end{array}\right\} \begin{array}{r} 21 - 20 = 1 \\ z = 1 \end{array} \left\{\begin{array}{r} 42,\ 63,\ 84 \\ 2,\ \ 3,\ \ 4 \end{array}\right\} \begin{array}{r} 84 - 20 = 64 \\ z = 4. \end{array}$$

(35.) *Ex.* 7. $39z - z^3 = 70$. Here $\dfrac{z^3 + 70}{z} = \dfrac{39}{1}$

$$\left.\begin{array}{r} z^3 + 70 = 39,\ 78 \\ z = 1,\ \ 2 \end{array}\right\} \begin{array}{r} 78 - 70 = 8 \\ z = 2 \end{array} \left\{\begin{array}{r} 117,\ 156,\ 195 \\ 3,\ \ 4,\ \ 5 \end{array}\right\} \begin{array}{r} 195 - 70 = 125 \\ z = 5. \end{array}$$

(35.) *Ex.* 8. $\frac{7}{4}z - z^3 = \frac{3}{4}$. Here $\dfrac{z^3 + \frac{3}{4}}{z} = \dfrac{\frac{7}{4}}{1}$

$$\left.\begin{array}{r} z^3 + \frac{3}{4} = \frac{7}{4} \\ z = 1 \end{array}\right\} \begin{array}{r} \frac{7}{4} - \frac{3}{4} = 1 \\ z = 1 \end{array} \left\{\begin{array}{c} \frac{7}{8} \\ \frac{1}{2} \end{array}\right\} \begin{array}{r} \frac{7}{8} - \frac{3}{4} = \frac{1}{8} \\ z = \frac{1}{2}. \end{array}$$

Ex. 9. $9x^2 - x^3 = 100$. Here $\dfrac{100 + x^3}{x^2} = \dfrac{9}{1}$ Or,

$\dfrac{x^2}{100 + x^3} = \dfrac{1}{9}$

$$\left.\begin{array}{r} x^2 = 1,\ \ 4,\ \ 9,\ \ 16,\ \ 25 \\ 100 + x^3 = 9,\ 36,\ 81,\ 144,\ 225 \end{array}\right\} \begin{array}{l} 25 = 5^2 = x^2 \\ 225 - 100 = 125 = 5^3 = x^3. \end{array}$$

Thus: we ſee in how very eaſy a manner theſe cubic equations may be reſolved, which we formerly found could not be reduced by the general method of evolution, (33, 35) And, we proceed in the ſame manner, in reſolving deficient equations of all degrees.

48. Corol. 7. When an equation is propoſed, involving more than two variable quantities, or more than two different powers of the ſame variable quantity; by ſubſtituting for the ſum, difference, product, quotient, &c. of two or more of theſe quantities, the equation may be transformed into another, involving only two variable quantities, or two different powers of the ſame variable quantity.

EXAMPLE.

Suppoſe $6x+7y+8z=100$; to find all the values of x, y, z, in whole numbers.

Let $(6x+8z=)\ 2\times\overline{3x+4z}=2v$; then $2v+7y=100$, and $\frac{50-v}{y}=\frac{7}{2}$

$$\left.\begin{array}{r}50-v=7\\ y=2\end{array}\right\}\begin{array}{ll}v=43,\ 36,\ 29,\ 22,\ 15. & (\text{Dif. }7.)\\ y=2,\ 4,\ 6,\ 8,\ 10. & (\text{Do. }2.)\end{array}$$

Taking $v=43$; we have $3x+4z=43$, or $\frac{x}{43-4z}=\frac{1}{3}$, and hence,

$$\left.\begin{array}{r}x=1\\ 43-4z=43\end{array}\right\}\begin{array}{ll}x=1,\ 5,\ 9,\ 13. & (\text{Dif. }4.)\\ z=10,\ 7,\ 4,\ 1. & (\text{Do. }3.)\end{array}$$

Thus: when $y=2$

$x=1,\ 5,\ 9,\ 13$

$z=10,\ 7,\ 4,\ 1.$

Taking $v=36$ . . . $3x+4z=36$ . . . . $3x=\overline{9-z}\times4$ . . . $\frac{x}{9-z}=\frac{4}{3}$.

$$\left.\begin{array}{r}x=4\\ 9-z=3\end{array}\right\}\begin{array}{ll}x=4,\ 8. & (\text{Dif. }4.)\\ z=6.\ 3. & (\text{Do. }3.)\end{array}$$

Thus: when $y=4$

$x=4,\ 8$

$z=6,\ 3.$

Taking $v=29$ . . . $3x+4z=29$ . . . . $\dfrac{x}{29-4z}=\dfrac{1}{3}$

$$\left.\begin{array}{r} x=1,\ 2,\ 3 \\ 29-4z=3,\ 6,\ 9 \end{array}\right\} \begin{array}{ll} x=3,\ 7. & \text{(Dif. 4.)} \\ z=5,\ 2. & \text{(Do. 3.)} \end{array}$$

Thus: when $y=6$

$x=3,\ 7$

$z=5,\ 2.$

Taking $v=22$ . . . $3x+4z=22$ . . . $\dfrac{x}{11-2z}=\dfrac{2}{3}$

$$\left.\begin{array}{r} x=2 \\ 11-2z=3 \end{array}\right\} \begin{array}{ll} x=2,\ 6 & \text{(Dif. 4.)} \\ z=4,\ 1 & \text{(Do. 3.)} \end{array}$$

Thus: when $y=8$

$x=2,\ 6$

$z=4,\ 1.$

Taking $v=15$ . . . $3x+4z=15$ . . . $\dfrac{5-x}{z}=\dfrac{4}{3}$ $\therefore$

$$\left.\begin{array}{r} 5-x=4 \\ z=3 \end{array}\right\} \begin{array}{l} x=1 \\ z=3. \end{array}$$

Thus: when $y=10$

$x=1$

$z=3$

And theſe are all the whole numbers by which x, y, z, can be expounded *.

SECTION III. OF THE COMPOSITION AND RESOLUTION OF QUANTITY.

PROPOSITION XX.

49. The root of *any cubic binomial, involving a quadratic ſurd quantity*, may be determined by the reſolution of a COMMENSURATE cubic equation, wanting the ſecond term.

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* See Dodſon's Repoſitory, v. 2. Simpſon's Select Exerciſes and Algebra.

Let $R \pm \sqrt{S}$ repreſent any cubic binomial, involving the radical quantity $\sqrt{S}$; ſuppoſe $\sqrt[3]{R^2 - S} = Q$, and let $2B$ be the value of z in this *commenſurate* cubic equation, $z^3 - 3Qz = 2R$ (47); then will $B \pm \sqrt{B^2 - Q}$ be the cube root of the propoſed binomial $R \pm \sqrt{S}$.

Ex. 1. Suppoſe $R = 20$, and $S = 392$; then $Q = 2$, and the equation $z^3 - 6z = 40$, gives $2B = 4$ (47); whence $B \pm \sqrt{B^2 - Q} = 2 \pm \sqrt{2}$.

Ex. 2. Suppoſe $R = 25$, and $S = 968$; then $Q = -7$, and the equation $z^3 + 21z = 50$, gives $2B = 2$ (47), whence, $B \pm \sqrt{B^2 - Q} = 1 \pm \sqrt{8}$.

Ex. 3. Suppoſe $R = 18$, and $S = 325$; then $Q = -2$, and the equation $z^3 + 12z = 288$, gives $2B = 6$ (47), whence $B \pm \sqrt{B^2 - Q} = \frac{3}{2} \pm \sqrt{3\frac{1}{4}}$.

Ex. 4. Suppoſe $R = 10$, and $S = 108$; then $Q = -2$, and the equation $z^3 + 6z = 20$, gives $2B = 2$ (47), whence, $B \pm \sqrt{B^2 - Q} = 1 \pm \sqrt{3}$.

Ex. 5. Suppoſe $R = 135$, and $S = 18152$; then $Q = -3$, and the equation, $z^3 + 9z = 270$, gives $2B = 6$ (47), whence $B \pm \sqrt{B^2 - Q} = 3 \pm \sqrt{12}$.

Ex. 6. Suppoſe $R = 68$, and $S = 4374$; then $Q = 5\sqrt[3]{2}$, and the equation, $z^3 - 15\sqrt[3]{2} \times z = 136$, by ſuppoſing $v = z \times \sqrt[3]{2}$, becomes $v^3 - 30v = 272$, which gives $v = 8$ (47); and therefore, $B \pm \sqrt{B^2 - Q} = \dfrac{4 - \sqrt{6}}{\sqrt[3]{2}}$.

Ex. 7. Suppoſe $R = 3$, and $S = -\dfrac{100}{27}$; then $Q = \dfrac{7}{3}$, and the equation, $z^3 - 7z = 6$, gives $2B = 3$ (47), whence $B \pm \sqrt{B^2 - Q} = \frac{3}{2} \pm \frac{1}{2}\sqrt{-\frac{1}{3}}$.

Ex. 8. Suppoſe $R = -10$, and $S = -243$; then $Q = 7$, and the equation, $21z - z^3 = 20$, gives $2B = 1$, as alſo $2B = 4$ (47); whence, taking $2B = 1$, we have $B \pm \sqrt{B^2 - Q} = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$, where $\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, is the root of $-10 - \sqrt{-243}$, and $\frac{1}{2}-\frac{3}{2}\sqrt{-3}$, that of $-10+\sqrt{-243}$; and taking $2B=4$, we have $B\pm\sqrt{B^2-Q}=2\pm\sqrt{-3}$.

Scholium.

50. As, by this propoſition, the root of *any* cubic binomial, may be determined; ſo thereby are we enabled to bring to a *direct ſolution*, all affected cubic equations whatſoever; (32, 33, 34, 35, 36.) Let $z^3-3qz-2r=0$, repreſent any affected cubic equation wanting the ſecond term, and let $m\pm\sqrt{n}$ repreſent the cube root of the binomial $r\pm\sqrt{r^2-q^3}$; then, ſince $z=2m$ (32), or $z-2m=0$, if we divide $z^3-3qz-2r$ by $z-2m$, we ſhall obtain this quadratic equation $z^2+2mz+\overline{4m^2-3q}=0$, which being reduced, (26), gives $z=-m\pm\sqrt{3q-3m^2}$ *admitting* affirmative and negative quantities: But $(y\times v=\overline{m+\sqrt{n}}\times\overline{m-\sqrt{n}}=)\ m^2-n=q$ (32), and therefore, by ſubſtitution, $z=-m\pm\sqrt{-3n}$. Thus: *admitting* affirmative and negative quantities; in the cubic equation, $z^3-3qz-2r=0$, we have

1. $z=2m$
2. $z=-m+\sqrt{-3n}$
3. $z=-m-\sqrt{-3n}$.

And therefore, in the general complete cubic equation (36), $x^3-3px^2+\overline{3p^2-3q}\times x-\overline{2r+p^3-3pq}=0$; ſince $x=p+z$, therefore,

1. $x=p+2m$
2. $x=p-m+\sqrt{-3n}$
3. $x=p-m-\sqrt{-3n}$.

51. In the application of this general equation to practiſe, the following obſervations may be attended to, (1.) That the third and fourth terms may be repreſented either as affirmative or negative, (37), by properly arranging theſe terms, (2.) That, if $p=0$, then

1. $x\ (=z)=2m$
2. $x\ (=z)=-m+\sqrt{-3n}$
3. $x\ (=z)=-m-\sqrt{-3n}$.

(3.) That, if $q=0$, then $m=\sqrt{n}$, (32); whence,

1. $z=2m\,(=\sqrt[3]{2r})$
2. $z=-m+\sqrt{-3m^2}$
3. $z=-m-\sqrt{-3m^2}$.

And therefore,
1. $x=p+2m$
2. $x=p-m+\sqrt{-3m^2}$
3. $x=p-m-\sqrt{-3m^2}$.

(4.) That, if $r=0$, then $m=0$, and $q=-n$; whence,

1. $z=0$
2. $z=\sqrt{3q}$
3. $z=-\sqrt{3q}$.

And therefore,
1. $x=p$
2. $x=p+\sqrt{3q}$
3. $x=p-\sqrt{3q}$.

Ex. 1. $x^3-12x^2+41x-42=0$.

Here $p=4$, $q=\frac{7}{3}$, $r=3$; and $r\pm\sqrt{r^2-q^3}=3\pm\sqrt{-\frac{100}{27}}$; whence $m\pm\sqrt{n}=\frac{3}{2}\pm\frac{1}{2}\sqrt{-\frac{1}{3}}$ (49), and therefore,

1. $x\,(=p+2m=4+3)=7$
2. $x\,(=p-m+\sqrt{-3n}=\frac{5}{2}+\frac{1}{2})=3$
3. $x\,(=p-m-\sqrt{-3n}=\frac{5}{2}-\frac{1}{2})=2$.

Ex. 2. $x^3+15x^2+84x-100=0$.

Here $p=-5$, $q=-3$, $r=135$, and $r\pm\sqrt{r^2-q^3}=135\pm\sqrt{18252}$; whence, $m\pm\sqrt{n}=3\pm\sqrt{12}$ (49), and therefore,

1. $x\,(=p+2m)=-5+6=1$
2. $x\,(=p-m+\sqrt{-3n})=-8+\sqrt{-36}$
3. $x\,(=p-m-\sqrt{-3n})=-8-\sqrt{-36}$.

Ex. 3. $x^3+9x^2-22x-120=0$.

Here $p=-3$, $q=\dfrac{49}{3}$, and $r=0$; whence,

1. $x\,(=p)=-3$
2. $x\,(=p+\sqrt{3q})=-3+7=4$,
3. $x\,(=p-\sqrt{3q})=-10$.

Ex. 4. $x^3 - 9x^2 + 27x - 35 = 0$.

Here $p = 3$, $q = 0$, $r = 4$; whence, $2m = \sqrt[3]{2r} = 2$, and therefore,

$1.\ x\ (= p + 2m) = 5$

$2.\ x\ (= p - m + \sqrt{-3m^2}) = 2 + \sqrt{-3}$

$3.\ x\ (= p - m - \sqrt{-3m^2}) = 2 - \sqrt{-3}$.

Ex. 5. $x^3 - 6x^2 - 9x + 54 = 0$.

Here $p = 2$, $q = 7$, $r = -10$, and $r \pm \sqrt{r^2 - q^3} = -10 \pm \sqrt{-243}$; whence, $m \pm \sqrt{n} = \frac{1}{2} \pm \frac{3}{2}\sqrt{-3}$, as alſo $m \pm \sqrt{n} = 2 \pm \sqrt{-3}$ (49), and therefore, taking the firſt value of $m \pm \sqrt{n}$

$1.\ x\ (= p + 2m) = 3$

$2.\ x\ (= p - m + \sqrt{-3n}) = 6$

$3.\ x\ (= p - m - \sqrt{-3n}) = -3$

Or, taking the ſecond value of $m \pm \sqrt{n}$

$1.\ x\ (= p + 2m) = 6$

$2.\ x\ (= p - m + \sqrt{-3n}) = 3$

$3.\ x\ (= p - m - \sqrt{-3n}) = -3$.

PROPOSITION XXI.

52. *Admitting* affirmative and negative quantities; and ſuppoſing affected equations of *all degrees* to be produced by a multiplication of binomial factors. Every affected equation will have as many roots as there are units in the exponent of the higheſt power of the variable quantity; and, if the terms of the equation are alternately affirmative and negative, the roots will be all affirmative; but, if the terms are all affirmative, the roots will be all negative: Moreover, the coefficient of the ſecond term, will be the ſum of all the roots with their ſigns changed; the coefficient of the third term, the ſum of all the products that can be formed, by taking the roots two and two under their proper ſigns; the coefficient of the fourth term, the ſum of all the products that can be formed by taking the roots three and three with their ſigns changed; and ſo on, the roots in all odd places, preſerving their ſigns, and in all even places, changing their ſigns. And as to the laſt term, it is always the product of all the roots.

This propoſition, which is proved by induction, was firſt given by Mr Harriot, the great improver of this part of the analytic art *.

Let x repreſent any variable quantity, and let a, b, c, d, e, &c. repreſent any conſtant quantities; if then we ſuppoſe, that x can be interpreted by any of theſe conſtant quantities, the product $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \overline{x-e}$, &c. will evidently vaniſh, or be equal to nothing: And, hence may be derived affected equations of all degrees.

Thus: Suppoſe $\overline{x-a} \times \overline{x-b} = 0$.

Then $x^2 \left.\begin{matrix}-a\\-b\end{matrix}\right\} x + ab = 0$, a quadratic equation.

Alſo, Suppoſing $\overline{x-a} \times \overline{x-b} \times \overline{x-c} = 0$,

Then $x^3 \left.\begin{matrix}-a\\-b\\-c\end{matrix}\right\} x^2 \left.\begin{matrix}+ab\\+ac\\+bc\end{matrix}\right\} x - abc = 0$, a cubic equation.

In like manner, ſuppoſing $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$

Then $x^4 \left.\begin{matrix}-a\\-b\\-c\\-d\end{matrix}\right\} x^3 \left.\begin{matrix}+ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{matrix}\right\} x^2 \left.\begin{matrix}-abc\\-abd\\-bcd\end{matrix}\right\} x + abcd = 0$

A biquadratic equation.

And ſo on, for equations of higher degrees.

Here the ſeveral particulars mentioned in this propoſition, appear by inſpection: And the ſame will be evident, ſuppoſing the ſigns of the quantities, a, b, c, d, &c. in the binomial factors, to be anyhow changed.

53. Corol. 1. Since the coefficient of the ſecond term of every affected equation, is the ſum of all the roots with their ſigns changed;

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* See Harriot's Praxis Artis Analyticæ, publiſhed by Mr Walter Warner, an. 1631; as alſo Dr Wallis's Algebra, Sir Iſaac Newton's Arithmetica Univerſalis, and Maclaurin's Algebra.

changed; therefore, in an affected equation, if the ſum of all the affirmative roots taken together, be equal to the ſum of all the negative roots taken together, the ſecond term will vaniſh. And conſequently, in an affected equation, wanting the ſecond term, the ſum of all the affirmative roots taken together, muſt be equal to that of all the negative roots taken together.

54. Corol. 2. In every affected equation, the coefficients of the third, fourth, fifth, &c. terms, are always **Divisors** of the laſt term.

55. Corol. 3. Since, in every affected equation, the laſt term is the product of all the roots; it is therefore evident, that every *commenſurate* affected equation, will have its roots among the diviſors, ſimple and compound, of the laſt term.

56. Corol. 4. Since unity is the leaſt ſimple diviſor of any quantity, therefore, if we divide any quantity by its leaſt diviſor, greater than unit, the quotient by its leaſt diviſor, greater than unit, and continue this proceſs until there be no remainder; it is evident, if we adjoin unity to theſe diviſors, we ſhall have all the ſimple diviſors of the quantity propoſed: And as for the compound diviſors, they are nothing more than the products of the ſimple diviſors taken two and two, three and three, four and four, &c.

57. Corol. 5. If therefore, we reſolve the laſt term of any commenſurate affected equation, into all its diviſors, ſimple and compound; we ſhall obtain the roots of this equation, ſuppoſing all the terms brought to one ſide, by obſerving what diviſors, taken affirmatively or negatively, and ſubſtituted inſtead of the variable quantity, make all the terms to vaniſh.

Ex. 1. $x^3 - 2x^2 - 5x + 6 = 0$,
Diviſors, 1, 2, 3, 6. Roots, 1, -2, 3.

Ex. 2. $x^3 - 2x^2 - 33x + 90 = 0$,
Diviſors, 1, 2, 3, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90.
Roots, 3, -5, 6.

Ex. 3. $x^4 - 4x^3 - 19x^2 + 160x - 120 = 0$,
Diviſors, 1, 2, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120. Roots, 2, 3, 4, -5.

58 Corol. 6. In any complete affected equation, involving both affirmative and negative roots, which may be known by the

ſigns of the terms; (ſuppoſing none of the roots to be imaginary or impoſſible quantities); the affirmative roots may be rendered negative, and the negative roots affirmative, by changing the ſigns of the alternate terms of the equation.

Ex. 1. In the equation, $x^3 - 2x^2 - 5x + 6 = 0$, the roots are, $1, -2, 3$; and, in the equation, $x^3 + 2x^2 - 5x - 6 = 0$, the roots are, $-1, 2, -3$ (57).

Ex. 2. In the equation, $x^3 - 2x^2 - 33x + 90 = 0$, the roots are, $3, -5, 6$; and, in the equation, $x^3 + 2x^2 - 33x - 90 = 0$, the roots are, $-3, 5, -6$ (57).

Ex. 3. In the equation, $x^4 - 4x^3 - 19x^2 + 160x - 120 = 0$, the roots are, $2, 3, 4, -5$; and, in the equation, $x^4 + 4x^3 - 19x^2 - 160x - 120 = 0$, the roots are, $-2, -3, -4, 5$ (57.)

Ex 4. In the equation, $x^3 + 12x^2 + 41x + 42 = 0$, the roots are, $-2, -3, -7$; and, in the equation, $x^3 - 12x^2 + 41x - 42 = 0$, the roots are, $2, 3, 7$ (57.)

59. Corol. 7. In every complete affected equation, free from fractions, radical quantities, and imaginary roots, there are ſo many affirmative roots, as there are changes of the ſigns in a continual ſeries, from + to —, and from — to +; and for every ſucceſſion of the ſame ſigns, there are negative roots, (57, 58.)

60. Corol. 8. When there are any terms wanting in affected equations, we may ſupply the deficient places with cyphers *.

PPOPOSITION XXII.

61. The root (x) of any equation whatſoever, may be increaſed and diminiſhed, multiplied and divided, by any given quantity, (a), taken at pleaſure.

In any equation, involving a variable quantity x, if we ſuppoſe $x = y + a$, or, $y = x - a$, and ſubſtitute, it is evident, that a new equation will be produced, involving the variable quantity y, which exceeds the variable quantity x, by the given quantity a (3): And, in like manner, by ſuppoſing $y = x - a$, or, $x = y + a$, and ſubſtituting, we ſhall have a new equation, involving the variable quantity y, which falls ſhort of the variable quantity x, by the given quantity a (4). Alſo, if we ſuppoſe $y = ax$, or, $x = \frac{y}{a}$, and ſubſtitute, a new equation will be produced, involving the variable quantity y, a multiple of the variable quantity x, by the given quantity a: And, in like manner, by ſuppoſing $y = \frac{x}{a}$ $(= \frac{1}{a} \times x)$, or, $x = ay$, and ſubſtituting, we ſhall have a new equation, involving the variable quantity y, a multiple of the variable quantity x, by the given quantity $\frac{1}{a}$ (5.)

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* For a full illuſtration of this propoſition, and its corollaries, ſee Sir Iſaac Newton's Arithmetica Univerſalis, with Dr Wilder's Annotations; as alſo Maclaurin's Algebra, and Saunderſon's Algebra.

Let $x^3 - px^2 + qx - r = 0$ repreſent a general affected cubic equation.

Suppoſe $(x + a = y$, or,$)$ $x = y - a$,

$$\text{(A)}\quad y^3 \left.\begin{matrix}-3a\\-p\end{matrix}\right\} y^2 \left.\begin{matrix}+3a^2\\+2pa\\+q\end{matrix}\right\} y \left.\begin{matrix}-a^3\\-pa^2\\-qa\\-r\end{matrix}\right\} = 0.$$

Suppoſe $(x - a = y$, or,$)$ $x = y + a$,

$$\text{(B)}\quad y^3 \left.\begin{matrix}+3a\\-p\end{matrix}\right\} y^2 \left.\begin{matrix}+3a^2\\-2pa\\+q\end{matrix}\right\} y \left.\begin{matrix}+a^3\\-pa^2\\+qa\\-r\end{matrix}\right\} = 0.$$

Suppoſe $(y = ax$, or$)$, $x = \frac{y}{a}$

$$\frac{y^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0.$$

Or,

$$\text{(C)}\quad y^3 - apy^2 + a^2qy - a^3r = 0.$$

Suppoſe $(y = \frac{1}{a} \times x$, or$)$, $x = ay$,

$$a^3y^3 - a^2py^2 + aqy - r = 0.$$

Or,

$$\text{(D)}\quad y^3 - \frac{p}{a}y^2 + \frac{q}{a^2}\,y - \frac{r}{a^3} = 0.$$

62. Corol. 1. By increaſing and diminiſhing the root (x), of any affected equation, by a proper quantity (a), we may exterminate any of the intermediate terms of the equation. Thus: in the equation, (A), ſuppoſe $-3a-p=0$, or, $a=-\frac{1}{3}p$, and the ſecond term will diſappear; and, in the equation (B), ſuppoſing $3a-p=0$, or, $a=\frac{1}{3}p$, the ſecond term vaniſhes alſo in this caſe. Thus: alſo, by ſuppoſing the coefficient of the third term equal to nothing, and reſolving the quadratic equation involving a, we may exterminate the third term. In general, by increaſing and diminiſhing the root of any affected equation, by a proper quantity (a); it will appear, that the ſecond, third, fourth, fifth, *&c.* terms, may be exterminated by the reſolution of a ſimple, an affected quadratic, affected cubic, affected biquadratic, *&c.* equation.

63. Corol. 2. From the laſt corollary, we have this general rule for exterminating the ſecond term of any affected equation, *viz.* Divide the coefficient of the ſecond term, by the exponent of the power of the equation, and increaſe or diminiſh the root, by this quotient, according as the ſecond term is affirmative or negative (61.)

That is, in the equation $x^m \pm P x^{m-1}$, *&c*, Suppoſe $y \mp \frac{P}{m} = x$.

Ex. 1. $x^2 - px - q = 0$.

Suppoſe $y + \frac{1}{2}p = x$

$$\left.\begin{array}{rl} x^2 = & y^2 + py + \frac{1}{4}p^2 \\ -px = & \quad -py - \frac{1}{2}p^2 \\ -\ q = & \qquad\qquad -\ q \end{array}\right\}$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$y^2 - \frac{1}{4}p^2 - q = 0.$$

Thus: $y = \sqrt{\frac{1}{4}p^2 + q}$, and $x\ (= \frac{1}{2}p + y) = \frac{1}{2}p + \sqrt{\frac{1}{2}p^2 + q}$ as before, (26.)

Ex. 2. $$\left.\begin{array}{r} x^3 - 3px^2 - 3q\ x - 2r \\ +3p^2x - p^3 \\ +3pq \end{array}\right\} = 0.$$

Suppoſe $y + p = x$, then $y^3 - 3qy - 2r = 0$.

64. Corol. 3. From the equations C and D, of this propoſition, it appears, that to multiply or divide the root (x), of any affected equation, by a given quantity a, we have only to range the terms, ſupplying deficient places with cyphers, and then multiplying or dividing the terms of the equation, by the terms of the progreſſion, $1, a, a^2, a^3, a^4, a^5$, &c.

Ex. 1. Suppoſe we were to multiply the roots of this equation by 2, *viz.* $x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$,

$$\begin{array}{cccccc} x^4 & +4x^3 & -19x^2 & -106x & -120 & =0 \\ 1 & 2 & 4 & 8 & 16 & \\ \hline y^3 & +8y^3 & -76y^2 & -848y & -1920 & =0, \end{array}$$

Where $y = 2x$.

Ex. 2. Let the roots of this equation, $x^3 - 3x + 1 = 0$, be multiplied by 3.

$$\begin{array}{ccccc} x^3 \ldots & 0x^2 \ldots & -3x & +1 & =0 \\ 1 & 3 & 9 & 27 & \\ \hline y^3 & & -27y & +27 & =0, \end{array}$$ Where $y = 3x$.

Ex. 3. Let the roots of this equation $x^4 + 2ax^3\sqrt{2} + 8abx^2 - a^3x\sqrt{8} - 2a^2b^2 = 0$ be divided by $\sqrt{2}$.

$$\begin{array}{cccccc} x^4 & +2ax^3\sqrt{2} & +8abx^2 & -a^3x\sqrt{8} & -2a^2b^2 & =0 \\ 1 & \sqrt{2} & 2 & \sqrt{8} & 4 & \\ \hline y^4 & +2ay^3 & +4aby^2 & -a^3y & -\frac{1}{2}a^2b^2 & =0 \end{array}$$

Where $y = \frac{x}{\sqrt{2}}$.

Ex. 4. Let the roots of this equation $x^3 - 84x + 160 = 0$, be divided by 2.

$$\begin{array}{ccccc} x^3 \ldots & 0x^2 \ldots & -84x & +160 & =0 \\ 1 & 2 & 4 & 8 & \\ \hline \end{array}$$

$y^3 - 21y + 20 = 0$, Where $y = \frac{x}{2}$.

65. Corol 4. By this propoſition, and the laſt corollary, we may change the root x of any equation, into its reciprocal, by ſuppoſing $y = \frac{1}{x}$, or $x = \frac{1}{y}$. Thus: in the equation.

$$x^4 - p\,x^3 + q\,x^2 - r\,x + s = 0,$$

$$\text{let } x = \frac{1}{y},$$

$$\text{Then } \frac{1}{y^4} - \frac{p}{y^3} + \frac{q}{y^2} - \frac{r}{y} + s = 0$$

$$\text{Or, } y^4 - \frac{r}{s}\,y^3 + \frac{q}{s}\,y^2 - \frac{p}{s}\,y + \frac{1}{s} = 0.$$

Where the order of the coefficients appears to be inverted by this operation.

66. Corol. 5. By increaſing and diminiſhing the root of an equation, by a proper quantity, we may diminiſh the abſolute term, and thus render the ſolution of commenſurate equations more ſimple and eaſy (57.)

Proposition XXIII.

67. In the general equation $A + Bx + Cx^2 + Dx^3 +$, &c. $= a + b\,x + c\,x^2 + d\,x^3 +$, &c. where the ſame powers of the ſame *variable quantity* (x), are found on both ſides of the equation: If we ſuppoſe the quantity A, to be equal to the quantity a; the coefficients of the ſame powers of the variable quantity (x), will all be equal.

68. Corollary. When therefore the quantity A is *actually* equal to the quantity a, the value of the expreſſion will be determined univerſally: And, when we *aſſume* the quantity A, equal to the quantity a, the value of the expreſſion will be determined agreeable to this ASSUMPTION ONLY.

Ex. 1.

Suppoſe $2\,A\,x + 2\,B\,x^2 + 2\,C\,x^3 + 2\,D\,x^4 + 2\,E\,x^5 +$, &c. $= 2\,A\,x + \overline{4B - A} \times x^2 + \overline{8\,C - 4B} \times x^3 + \overline{16D - 12C + B} \times x^4 + \overline{32\,E - 32\,D + 6\,C} \times x^5 +$, &c.

Here

$$\begin{aligned}
\text{Here } 4B - A &= 2B \;.\;.\;.\;.\;.\; B = \tfrac{1}{2}A \\
8C - 4B &= 2C \;.\;.\;.\;.\; C = \tfrac{1}{3}A \\
16D - 12C + B &= 2D \;.\;.\;.\;.\;.\; D = \tfrac{1}{4}A \\
32E - 32D + 6C &= 2E \;.\;.\;.\;.\;.\; E = \tfrac{1}{5}A, \text{ \&c. \&c.}
\end{aligned}$$

And, thus it appears, that either of the above expreſſions is equivalent to this ſeries, *viz.*

$$2A \times \overline{x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 +}, \text{ \&c.}$$

Ex. 2.

Suppoſe $\frac{m}{n} + \frac{m}{n}Ax + \frac{m}{n}Bx^2 + \frac{m}{n}Cx^3 + \frac{m}{n}Dx^4 +$, &c. $= A + \overline{2B + A} \times x + \overline{3C + 2B} \times x^2 + \overline{4D + 3C} \times x^3 + \overline{5E + 4D} \times x^4 +$, &c.

Here aſſuming $A = \frac{m}{n}$,

$$\begin{aligned}
\text{Then } 2B + A &= \frac{m}{n}A \;.\;.\;.\;.\; B = \frac{m-n}{2n}A \\
3C + 2B &= \frac{m}{n}B \;.\;.\;.\;.\; C = \frac{m-2n}{3n}B \\
4D + 3C &= \frac{m}{n}C \;.\;.\;.\;.\; D = \frac{m-3n}{4n}C \\
5E + 4D &= \frac{m}{n}D \;.\;.\;.\;.\; E = \frac{m-4n}{5n}D \\
\text{\&c.} & \qquad\qquad\qquad \text{\&c.}
\end{aligned}$$

And thus, upon the ſuppoſition, that $A = \frac{m}{n}$, the value of either of the above expreſſions is equivalent to this ſeries, *viz.*

$$\frac{m}{n} + \frac{m-n}{2n}Ax + \frac{m-2n}{3n}Bx^2 + \frac{m-3n}{4n}Cx^3 + \frac{m-4n}{5n}Dx^4 +, \text{ \&c.}$$

SCHOLIUM.

69. This propoſition is the foundation of all Sir Iſaac Newton's diſcoveries in Mathematical Analyſis: And upon it depends the whole of the doctrine of infinite ſeries, approximations, &c.

PROPOSITION XXIV.

70. Suppoſing m, n, to repreſent any affirmative integral numbers, if the quantity $v^{\frac{m}{n}} - z^{\frac{m}{n}}$, be divided by the quantity $v-z$, the quotient will be equal to

$$v^{\frac{m}{n}-1} \times \frac{1+\frac{z}{v}+\left.\overline{\frac{z}{v}}\right|^{2}+\left.\overline{\frac{z}{v}}\right|^{3}\ (m)}{1+\left.\overline{\frac{z}{v}}\right|^{\frac{m}{n}}+\left.\overline{\frac{z}{v}}\right|^{\frac{2m}{n}}+\left.\overline{\frac{z}{v}}\right|^{\frac{3m}{n}}\ (n)}.$$

This is Mr Landen's Theorem, upon which he founds his RESIDUAL ANALYSIS; and it is eaſily demonſtrated by diviſion.

71. Corol. 1. By ſuppoſing $v = z$, the quantity $\frac{v^{\frac{m}{n}} - z^{\frac{m}{n}}}{v-z}$ vaniſhes altogether, and the ſeries becomes $\frac{m}{n} \times v^{\frac{m}{n}-1}$: This quantity $\frac{m}{n} \times v^{\frac{m}{n}-1}$ is thereſore called the LIMIT of the fraction $\frac{v^{\frac{m}{n}} - z^{\frac{m}{n}}}{v-z}$, when both the terms are ſuppoſed to vaniſh.

72. Corol. 2. Let the variable quantity x, be ſuppoſed to be increaſed by the quantity w; then, when x becomes $x+w$, it is evident, that $x^{\frac{m}{n}}$ will become $\left.\overline{x+w}\right|^{\frac{m}{n}}$, the increment acquired by $x^{\frac{m}{n}}$, being $\left.\overline{x+w}\right|^{\frac{m}{n}} - x^{\frac{m}{n}}$, when that acquired by x, is (w, or) $\overline{x+w} - x$. If therefore, we ſuppoſe, $v = x + w$, and $z = x$,

then by ſubſtitution $\frac{\overline{x+w}|^{\frac{m}{n}} - x^{\frac{m}{n}}}{\overline{x+w}-x} = \overline{x+w}|^{\frac{m}{n}-1} \times$

$$\frac{1+\frac{x}{x+w}+\overline{\frac{x}{x+w}}|^{2}+\overline{\frac{x}{x+w}}|^{3}\ (m)}{1+\overline{\frac{x}{x+w}}|^{\frac{m}{n}}+\overline{\frac{x}{x+w}}|^{\frac{2m}{n}}+\overline{\frac{x}{x+w}}|^{\frac{3m}{n}}\ (n)}$$

: And, if we ſuppoſe w to vaniſh, the quantity $\frac{\overline{x+w}|^{\frac{m}{n}} - x^{\frac{m}{n}}}{\overline{x+w}-x}$ vaniſhes altogether, and the ſeries becomes $\frac{m}{n} \times x^{\frac{m}{n}-1}$, which, therefore, is the LIMIT (71) of the quantity $\frac{\overline{x+w}|^{\frac{m}{n}} - x^{\frac{m}{n}}}{\overline{x+w}-x}$: Or, which is the ſame, the ratio $\frac{m}{n} \times x^{\frac{m}{n}-1} : 1$, is according to Sir Iſaac Newton, the ULTIMATE RATIO of the increments $\overline{x+w}|^{\frac{m}{n}} - x^{\frac{m}{n}}$ and $\overline{x+w}-x$.

73. Corol. 3. Suppoſe $v=1+Q$ and $z=1+R$; then by ſubſtitution, $\left(\frac{\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}}}{\overline{1+Q}-\overline{1+R}} =\right) \frac{\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}}}{Q-R}$

$$= \overline{1+Q}|^{\frac{m}{n}-1} \times \frac{1+\frac{1+R}{1+Q}+\overline{\frac{1+R}{1+Q}}|^{2}+\overline{\frac{1+R}{1+Q}}|^{3}\ (m)}{1+\overline{\frac{1+R}{1+Q}}|^{\frac{m}{n}}+\overline{\frac{1+R}{1+Q}}|^{\frac{2m}{n}}+\overline{\frac{1+R}{1+Q}}|^{\frac{3m}{n}}\ (n)}.$$

H Here

Here ſuppoſing $1+Q=1+R$, the quantity $\frac{\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}}}{Q-R}$ vaniſhes altogether, and the ſeries becomes $\frac{m}{n} \times \overline{1+Q}|^{\frac{m}{n}-1}$.

74. Corol. 4. Suppoſe $\overline{1+Q}|^{\frac{m}{n}} = 1 + AQ + BQ^2 + CQ^3 + DQ^4 +$, &c. then $\overline{1+R}|^{\frac{m}{n}} = 1 + AR + BR^2 + CR^3 + DR^4 +$, &c. And therefore, $\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}} = A \times \overline{Q-R} + B \times \overline{Q^2-R^2} + C \times \overline{Q^3-R^3} + D \times \overline{Q^4-R^4} +$, &c. from whence we have $\frac{\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}}}{Q-R} = A + B \times \overline{Q+R} + C \times \overline{Q^2+QR+R^2} +$, &c. univerſally.

Here, ſuppoſing $Q=R$, the quantity $\frac{\overline{1+Q}|^{\frac{m}{n}} - \overline{1+R}|^{\frac{m}{n}}}{Q-R}$ vaniſhes altogether, and the ſeries becomes $A + 2BQ + 3CQ^2 + 4DQ^3 +$, &c. $= \frac{m}{n} \times \overline{1+Q}|^{\frac{m}{n}-1}$ (73.)

75. Corol. 5. Since then $\frac{m}{n} \times \overline{1+Q}|^{\frac{m}{n}-1} = A + 2BQ + 3CQ^2 + 4DQ^3 +$, &c. therefore, multiplying both ſides by $1+Q$ or $\overline{1+Q}|^1$ we have $\frac{m}{n} \times \overline{1+Q}|^{\frac{m}{n}} = A + \overline{2B+A} \times Q + \overline{3C+2B} \times Q^2 + \overline{4D+3C} \times Q^3 +$, &c. $= \frac{m}{n} + \frac{m}{n} AQ + \frac{m}{n} BQ^2 + \frac{m}{n} CQ^3 +$, &c. by ſuppoſition (74): whence,

taking $A = \frac{m}{n}$, it is, (as in 68. *Ex.* 2.) $\overline{1+Q}|^{\frac{m}{n}} = 1 + \frac{m}{n} Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n} Q^3 +$, &c.

76. Corol. 6. It appears from the laſt corollary, that $\overline{1+Q}|^{\frac{m}{n}} = 1 + \frac{m}{n} Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n} Q^3$ +, &c. therefore, $\left(\frac{1}{\overline{1+Q}|^{\frac{m}{n}}} =\right) \overline{1+Q}|^{-\frac{m}{n}}$ (10) =

$$\frac{1}{1 + \frac{m}{n} Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n} Q^3 +, \&c.}$$

$= 1 - \frac{m}{n} Q - \frac{m}{n} \cdot \frac{-m-n}{2n} Q^2 - \frac{m}{n} \cdot \frac{-m-n}{2n} \cdot \frac{-m-2n}{3n} Q^3 -$, &c. by actual diviſion, (7, 8.) The ſeries, in this caſe, as is obvious, can never terminate.

77. Corol. 7. Thus: then it appears, ſuppoſing $\frac{m}{n}$ to be any number, integral or fractional, affirmative or negative; that $\overline{1+Q}|^{\frac{m}{n}} = 1 + \frac{m}{n} Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n} Q^3 +$, &c. which is Sir Iſaac Newton's famous binomial theorem, inveſtigated univerſally, according to Mr Landen's method *(75,76).

78. Corol. 8. Let P + PQ repreſent any binomial, then, ſince $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} \times \overline{1+Q}|^{\frac{m}{n}}$ (5), therefore, by the laſt co-

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* Diſcourſe concerning the Riſidual Analyſis, pages 6th and 7th.

rollary (77), $\overline{P+PQ}|^{\frac{m}{n}} = \overset{A}{P^{\frac{m}{n}}} + \overset{B}{\frac{m}{n}AQ} + \overset{C}{\frac{m-n}{2n}BQ} + \overset{D}{\frac{m-2n}{3n}CQ} + \overset{E}{\frac{m-3n}{4n}DQ} +$, *&c.* Here P is the firſt term, Q ſecond divided by the firſt, $\frac{m}{n}$, the exponent, and A, B, C, D, E, *&c.* the foregoing terms with their proper ſigns; and this is the general formula, called the *Binomial Theorem*, as given by the illuſtrious inventor, Sir Iſaac Newton, in his letter to Mr Oldenburgh, 13th June 1676; which letter, was tranſmitted by Mr Oldenburgh to M. Leibnitz, on the 26th of June 1676 *. By this moſt excellent theorem, we can multiply and divide, involve and evolve all quantities, whether terminate or interminate. We ſhall illuſtrate this theorem by ſome eaſy examples.

Ex. 1. Suppoſe we were to raiſe 18 to the 4th power. Here $18 = 10 + 8$; and ſo $P = 10$, $Q = \frac{8}{10} = \frac{4}{5}$, and $\frac{m}{n} = \frac{4}{1}$, or $m = 4$, and $n = 1$.

$$P^{\frac{m}{n}} \ldots\ldots\ldots\ldots\ldots\ldots = 10000 = A$$

$$\frac{m}{n}AQ = 4 \times 10000 \times \frac{4}{5} \ldots\ldots\ldots\ldots = 32000 = B$$

$$\frac{m-n}{2n}BQ = \frac{3}{2} \times 32000 \times \frac{4}{5} \ldots\ldots\ldots = 38400 = C$$

$$\frac{m-2n}{3n}CQ = \frac{2}{3} \times 38400 \times \frac{4}{5} \ldots\ldots\ldots = 20480 = D$$

$$\frac{m-3n}{4n}DQ = \frac{1}{4} \times 20480 \times \frac{4}{5} \ldots\ldots\ldots = 4096 = E$$

$$\overline{P+PQ}|^{\frac{m}{n}} = \overline{18}|^{4} \ldots\ldots\ldots\ldots = 104976$$

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* See Commercium Epiſtolicum, p. 49.—57; as alſo Dr Wallis's Algebra.

And the ſame may be derived by dividing 18 into any other two parts. Thus: Since $18 = 6 + 12$; therefore, $P = 6$, $Q = \frac{12}{6} = 2$

$$P^{\frac{m}{n}} = \overline{6}|^{4} \ldots\ldots\ldots\ldots = 1296 = A$$

$$\frac{m}{n} AQ \ldots\ldots\ldots\ldots = 10368 = B$$

$$\frac{m-n}{2n} BQ \ldots\ldots\ldots\ldots = 31104 = C$$

$$\frac{m-2n}{3n} CQ \ldots\ldots\ldots\ldots = 41472 = D$$

$$\frac{m-3n}{4n} DQ \ldots\ldots\ldots\ldots = 20736 = E$$

$$\overline{P+PQ}|^{\frac{m}{n}} = \overline{18}|^{4} \ldots\ldots\ldots = 104976$$

Ex. 2. To extract the ſquare root of the number 2. Here $2 = 1 + 1$; whence, $P = 1$, $Q = 1$, and $\frac{m}{n} = \frac{1}{2}$, or $m = 1$, and $n = 2$.

$$P^{\frac{m}{n}} \ldots\ldots\ldots\ldots = 1 = A$$

$$\frac{m}{n} AQ = \frac{1}{2} \times 1 \ldots\ldots\ldots\ldots = \frac{1}{2} = B$$

$$\frac{m-n}{2n} BQ = -\frac{1}{4} \times \frac{1}{2} \ldots\ldots = -\frac{1}{2.4} = C$$

$$\frac{m-2n}{3n} CQ = -\frac{3}{6} \times -\frac{1}{2.4} \ldots\ldots = \frac{1.3}{2.4.6} = D$$

$$\frac{m-3n}{4n} DQ = -\frac{5}{8} \times \frac{1.3}{2.4.6} \ldots\ldots = -\frac{1.3.5}{2.4.6.8} = E$$

&c. &c.

Here the law of the ſeries is obvious; and $\sqrt{2} = 1\tfrac{1}{2} - \frac{1}{2.4} + \frac{1.3}{2.4.6} - \frac{1.3.5}{2.4.6.8} + \frac{1.3.5.7}{2.4.6.8.10} -$ &c.

Ex. 3. To extract the ſquare root of $a^2 - x^2$. Here $P = a^2$, $Q = -\frac{x^2}{a^2}$, $\frac{m}{n} = \frac{1}{2}$, or $m = 1$, and $n = 2$.

$$P^{\frac{m}{n}} = \overline{a^2}|^{\frac{1}{2}} \quad \ldots \ldots \ldots \ldots \quad = a = A$$

$$\frac{m}{n} A Q = \frac{1}{2} a \times -\frac{x^2}{a^2} \quad \ldots \ldots \quad = -\frac{x^2}{2.a} = B$$

$$\frac{m-n}{2n} B Q = -\frac{1}{4} \times -\frac{x^2}{2.a} \times -\frac{x^2}{a^2} \;. = -\frac{x^4}{2.4.a^3} = C$$

$$\frac{m-2n}{3n} C Q = -\frac{3}{6} \times -\frac{x^4}{2.4.a^3} \times -\frac{x^2}{a^2} = -\frac{3.x^6}{2.4.6a^5} = D$$

&c. &c.

Here the law of the ſeries appears; and $\sqrt{a^2 - x^2}$, or $\overline{a^2 - x^2}|^{\frac{1}{2}}$ $= a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9} -$, &c.

Ex. 4. To determine the value of $\frac{ay}{\sqrt{a^2 - y^2}}$ in a ſeries. Here $\frac{ay}{\sqrt{a^2 - y^2}} = ay \times \overline{a^2 - y^2}|^{-\frac{1}{2}}$; whence, $P = a^2$, $Q = -\frac{y^2}{a^2}$, $\frac{m}{n} = -\frac{1}{2}$, or $m = -1$, and $n = 2$.

$$P^{\frac{m}{n}} = a^{2 \times -\frac{1}{2}} = a^{-1} \quad \ldots \ldots \ldots \quad = \frac{1}{a} = A$$

$$\frac{m}{n} A Q = -\frac{1}{2} \times \frac{1}{a} \times -\frac{y^2}{a^2} \quad \ldots \ldots \quad = \frac{y^2}{2.\,a^3} = B$$

$$\frac{m-n}{2n} B Q = -\frac{3}{4} \times \frac{y^2}{2.a^3} \times -\frac{y^2}{a^2} \quad .\,. \quad = \frac{3.\,y^4}{2.4.a^5} = C$$

$$\frac{m-2n}{3n} C Q = -\frac{5}{6} \times \frac{3.\,y^4}{2.4.\,a^5} \times -\frac{y^2}{a^2} \;.\,. = \frac{3.5.y^6}{2.4.6.\,a^7} = D$$

&c. &c.

Here the law of the ſeries is obvious; and $\overline{a^2-y^2}\big|^{-\frac{1}{2}} = \frac{1}{a} + \frac{y^2}{2a^3} + \frac{3y^4}{8a^5} + \frac{5y^6}{16a^7} +$, &c. and therefore,

$$\left(ay \times \overline{a^2-y^2}\big|^{-\frac{1}{2}} =\right) \frac{ay}{\sqrt{a^2-y^2}} = y + \frac{y^3}{2a^2} + \frac{3y^5}{8a^4} + \frac{5y^7}{16a^6} +, \&c.$$

Ex. 5. $\sqrt[3]{1-x^3} = \overline{1-x^3}\big|^{\frac{1}{3}} = 1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81} - \frac{10x^{12}}{243} - \frac{22x^{15}}{729} -$, &c.

Ex. 6. $\sqrt[3]{\frac{a^2}{\overline{a^2+x^2}\big|^2}} = a^{\frac{2}{3}} \times \overline{a^2+x^2}\big|^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{a^2}} \times \overline{1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6} +, \&c.}$

Ex. 7. $\sqrt[5]{a^2-x^2} = \overline{a^2-x^2}\big|^{\frac{1}{5}} = a^{\frac{2}{5}} \times \overline{1 - \frac{x^2}{5a^2} - \frac{2x^4}{25a^4} - \frac{6x^6}{125a^6} - \frac{21x^8}{625a^8} -, \&c.}$

79. Corol. 9. The theorem given in the laſt corollary, may be applied to any quantity, terminate or interminate, either by ſubſtitution, or by connecting the terms properly with the vinculum. And, hence may be derived Sir Iſaac Newton's two theorems for the reverſion of ſeries, and all others of the ſame kind. Thus:

Caſe I. Suppoſe $v = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 +$, &c.

Then $z = \frac{1}{A}v - \frac{B}{A^3}v^2 + \frac{2B^2 - AC}{A^5}v^3 + \frac{5ABC - A^2D - 5B^3}{A^7}v^4 + \frac{14B^4 - 21AB^2C + 6A^2BD + 3A^2C^2 - A^3E}{A^9}v^5 +$, &c.

Caſe II. Suppoſe $v = Az + Bz^3 + Cz^5 + Dz^7 +$, &c.

Then $z = \frac{1}{A}v - \frac{B}{A^4}v^3 + \frac{3B^2 - AC}{A^7}v^5 + \frac{8ABC - A^2D - 12B^3}{A^{10}}v^7$

$$+ \frac{55B^4 - 55AB^2C + 10A^2BD + 5A^2C^2 - A^3E}{A^{13}}v^9$$

$+$, &c.

Caſe III. Suppoſe $v = Az^{m} + Bz^{m+n} + Cz^{m+2n} + Dz^{m+3n} +$, &c. Then (ſuppoſing $u = \frac{v}{A}$) $z = u^{\frac{1}{m}}$

$$- \frac{B}{mA}u^{\frac{1+n}{m}} + \frac{\overline{m+1+2n}B^2 - 2mAC}{2m^2A^2}u^{\frac{1+2n}{m}}$$

$$\left.\begin{array}{r} -2m^2 + 9mn + 9n^2 + 3m + 6n + B^3 \\ + \frac{m+3n+1}{m^3A^3}BC \\ - \frac{D}{mA} \end{array}\right\} u^{\frac{1+3n}{m}} +, \text{\&c.}$$

Caſe IV. Suppoſe $av + bv^2 + cv^3 + dv^4 +$, &c. $=$ $gz + hz^2 + kz^3 + lz^4 +$, &c. Then $v = \frac{g}{a}v +$

$$\frac{h - bA^2}{a}v^2 + \frac{k - 2bAB - cA^3}{a}v^3 +$$

$$\frac{l - bB^2 - 2bAC - 3cA^2B - dA^4}{a}v^4 +, \text{\&c.}$$

Where A, B, C, D, &c. are the coefficients of the firſt, ſecond, third, &c. terms.

Ex. 1. Suppoſe $v = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 +$, &c.

Then (Caſe I.) $z = v + \frac{v^2}{1.2} + \frac{v^3}{1.2.3} + \frac{v^4}{1.2.3.4} + \frac{v^5}{1.2.3.4.5}$ $+$, &c.

Ex. 2. Suppoſe $v = z + \frac{1}{6}z^3 + \frac{3}{40}z^5 + \frac{5}{112}z^7 +$, &c.

Then (Caſe II.) $z = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} +$, &c.

Ex. 3. Suppoſe $v = z - \frac{a^2}{2z} + \frac{a^4}{6z^3} - \frac{a^6}{24z^5} +$, &c.

Then (Caſe III.) $z = v + \frac{a^2}{2v} - \frac{5a^4}{12v^3} + \frac{5a^6}{8v^5}$, &c.

Ex. 4. Suppoſe $v - \frac{1}{2}v^2 + \frac{1}{6}v^3 - \frac{1}{24}v^4$, &c. $= \frac{1}{2}z + \frac{1}{3}z^2 + \frac{1}{4}z^3 + \frac{1}{5}z^4$, &c.

Then (Caſe IV.) $v = \frac{1}{2}z + \frac{11}{24}z^2 + \frac{11}{24}z^3 + \frac{1381}{2880}z^4$, &c.

PROPOSITION XXV *.

80. In the ſingle equation $y^m = N$, ſuppoſe $\frac{A}{B}$ to be a near approximation to the value of y; then, by continually ſubſtituting the approximating quantities into this general formula

$$\frac{N + \overline{m-1} \times \overline{\frac{A}{B}}\Big|^{m}}{m \times \overline{\frac{A}{B}}\Big|^{m-1}}$$

The value of y may be determined to *any degree of accuracy*. Or, univerſally let $ay^m + by^{m-1} + cy^{m-2} + dy_m{}^{-3}$, &c. $= 0$, repreſent any affected equation; and, ſuppoſe $\frac{A}{B}$ to be a near approximation to the value of y; then, by continually ſubſtituting the approximating quantities into the following general formula, the value of y may be determined to *any degree of accuracy*.

GENERAL FORMULA.

$$\frac{\overline{m-1}\,aA^m + \overline{m-2}\,bA^{m-1}B + \overline{m-3}\,cA^{m-2}B^2 +, \&c.}{maA^{m-1}B + \overline{m-1}\,bA^{m-2}B^2 + \overline{m-2}\,cA^{m-3}B^3 +, \&c.}$$

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* See Maclaurin's Algebra, Simpſon's Algebra and Select Exerciſes, as alſo Emerſon's Algebra.

This propoſition is demonſtrated directly by Sir Iſaac Newton's Binomial Theorem, (78) *, and the general formula will reſolve any affected equation whatſoever. Thus: in the quadratic equation, $y^2 + by = c$, the approximating quantity is $\frac{A^2 + cB^2}{\overline{2A + bB} \times B}$; in the cubic equation, $y^3 + by^2 + cy = d$, the approximating quantity is $\frac{\overline{2A + bB} \times A^2 + dB^3}{\overline{3A + 2bAB + cB^2} \times B}$; in the biquadratic equation, $y^4 + by^3 + cy^2 + dy = c$, the approximating quantity is $\frac{\overline{3A^2 + 2bAB + cB^2} \times A^2 + cB^4}{\overline{4A^3 + 3bA^2B + 2cAB^2 + dB^3} \times B}$; and thus, from the general formula, you may derive the approximating theorem for any equation.

Ex. 1. Suppoſe $y^2 = 2$.

Here taking $\frac{A}{B} = 1$ $(= \frac{1}{1})$ for the firſt approach to the root; we have $y = \frac{3}{2}$ for a nearer approximation. *Again*, taking $\frac{A}{B} = \frac{3}{2}$, we have $y = \frac{17}{12}$ for a third approximation. Alſo, taking $\frac{A}{B} = \frac{17}{12}$, we have $y = \frac{3462}{2448}$ for a fourth approximation. And taking $\frac{A}{B} = \frac{3462}{2448}$, we ſhall obtain a fifth approximation, which being reduced, will give $y = 1.414213562378$, nearly.

Ex. 2. Suppoſe $y^2 = 12$.

Firſt approximation, $\frac{A}{B}$ $(= 3) = \frac{3}{1}$.

Second,——— $\frac{A}{B}$ $(= \frac{9 + 12}{6}) = \frac{7}{2}$.

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* See Mr Colſon's Commentary on Sir Iſaac Newton's Method of Fluxions, p. 186. — 191.

Third, —— $\frac{A}{B}\left(=\frac{49+48}{14\times 2}\right)=\frac{97}{28}$.

Fourth, —— $\frac{A}{B}\left(=\frac{\overline{97}|^2+12\times\overline{28}|^2}{194\times 28}\right)=\frac{18817}{5432}$.

Fifth, —— $\frac{A}{B}\left(=\frac{\overline{18817}|^2+12\times\overline{5432}|^2}{37634\times 5432}\right)=\frac{708158977}{204427888}=$ 3.4641016151377545 9, nearly.

Ex. 3. Suppoſe $y^3-2y=5$.

Here $a=1$, $b=0$, $c=-2$, and $d=5$.

Firſt approach, $\frac{A}{B}\ (=2)=\frac{2}{1}$.

Second, —— $\frac{A}{B}\left(=\frac{16+5}{10}\right)=\frac{21}{10}$.

Third, —— $\frac{A}{B}\left(=\frac{9261-2500}{5615}\right)=\frac{11761}{5615}$.

Fourth, —— $\frac{A}{B}\left(=\frac{2\times\overline{11761}|^3+5\times\overline{5015}|^3}{3\times\overline{11761}|^2\times 5615-2\times\overline{5615}|^3}\right)=$

$\frac{4138744325037}{1975957316495}=$ 2.094551481701, nearly.

Ex. 4. Suppoſe $y^3+pxy+p^2y-x^3-2p^3=0$, to determine the value of y in a converging ſeries, ſuppoſing x to be ſmall in compariſon of y.

Since x is ſuppoſed to be ſmall in compariſon of y, that the ſeries may converge; therefore, to determine the firſt approximation, ſuppoſe x to vaniſh, and we have $y^3+p^2y-2p^3=0$, which gives $y=p\ (57)=\frac{A}{B}$. Then, reducing the equation to a proper form, *viz.*

$$y^3+\overline{p^2+px}\times y-\overline{2p^3+x^3}\times y^0=0,$$

It appears directly, that $m=3$, $a=1$, $b=0$, $c=p^2+px$, $d=-2p^3-x^3$, $e=0$. Thus:

Firſt approach, $\frac{A}{B}\ (=p)=\frac{p}{1}$.

2d, —— $\frac{A}{B} = \frac{4p^3 + x^3}{4p^2 + px}$.

3d, —— $\frac{A}{B} = \frac{256p^9 + 96p^8x + 24p^7x^2 + 114p^6x^3 + 48p^5x^4 +, \&c.}{256p^8 + 160p^7x + 60p^6x^2 + 109p^5x^3 + 25p^4x^4 +, \&c.}$

$= p - \frac{1}{4}x + \frac{x^2}{64p} + \frac{131x^3}{512p^2} + \frac{509x^4}{16384p^3}$, &c.

81. Corol. 1. Although this propoſition is general, yet in reſolving literal equations (*Ex.* 4.) the following method will often be found more eaſy. In a literal affected equation, involving two variable quantities, x, y, aſſume $y = Ax^{n} + Bx^{n+r} + Cx^{n+s} + Dx^{n+t}$, &c. And, inſtead of y and its powers, in the propoſed equation, ſubſtitute Ax^n and its powers; then, to determine n, ſuppoſe the two leaſt exponents equal for an aſcending ſeries, or the two greateſt for a deſcending one: Subſtitute the value of n thus obtained, into all theſe exponents, and having taken the leaſt for an aſcending ſeries, or the greateſt for a deſcending one; ſubtract it from each of the reſt, and the remainders added to themſelves, and to one another, all *poſſible* ways, will give the values of r, s, t, &c. And as for the coefficients, A, B, C, D, &c they will be obtained by Prop. XX. (67, 68.)

Ex. 1. $p^4x^2 - p^4xy + x^6 - py^5 = 0$,

Here $p^4x^2 - p^4Ax^{n+1} + x^6 - pA^5x^{5n} = 0$.

Suppoſing $n + 1 = 2$, we have $n = 1$,

Exponents, 2, 2, 6, 5,

Remainders, 3, 4, which give the numbers, 3, 4, 6, 7, 8, 9, 10, &c. for r, s, t, &c.

Thus: For an aſcending ſeries, $y = Ax + Bx^4 + Cx^5 + Dx^7 +$, &c.

$$\left.\begin{aligned} p^4x^2 &= p^4x^2 \\ -p^4xy &= -p^4Ax^2 - p^4Bx^5 - p^4Cx^6 - p^4Dx^8 \\ +\ x^6 &= \ .\ .\ .\ .\ .\ +\ x^6 \\ -py^5 &= \ .\ .\ -pA^5x^5\ .\ -5pA^4Bx^8,\ \&c. \end{aligned}\right\} = 0.$$

Therefore, equating the homologous terms, (68).

$$p^4 - p^4 A = 0 \quad \ldots\ldots\ldots\ldots \quad A = 1$$

$$-p^4 B - p A^5 = 0 \quad \ldots\ldots\ldots\ldots \quad B = -\frac{1}{p^3}$$

$$-p^4 C + 1 \quad = 0 \quad \ldots\ldots\ldots\ldots \quad C = \frac{1}{p^4}$$

$$-p^4 D - 5 p A^4 B = 0 \quad \ldots\ldots\ldots\ldots \quad D = \frac{5}{p^6}$$

&c. *&c.*

Hence, $y = x - \frac{x^4}{p^3} + \frac{x^5}{p^4} + \frac{5x^7}{p^6}$, *&c.*

If we ſuppoſe $5n = 6$, or $n = 1\frac{1}{5}$, we ſhall have for a deſcending ſeries, $y = A x^{1^{1}} + B x^{-2\frac{3}{5}} + C x^{-2\frac{4}{5}} + D x^{-6\frac{2}{5}}$, *&c.* $= \frac{x^{1\frac{1}{5}}}{p^{\frac{1}{5}}} - \frac{p^{3\frac{3}{5}}}{5 x^{2\frac{3}{5}}} + \frac{p^{3\frac{4}{5}}}{5 x^{2\frac{4}{5}}}$, *&c.* viz. by proceeding as above.

Ex. 2. $y^3 + p^2 y + p x y - x^3 - 2 p^3 = 0$,

$$A^3 x^{3n} + p^2 A x^{n} + p A x^{n+1} - x^3 - 2 p^3 x^0 = 0.$$

Suppoſe $n = 0$, for an aſcending ſeries.

Series for r, s, t, &c. 1, 2, 3, 4, 5, *&c.*

And $y = A + B x + C x^2 + D x^3 + E x^4$, *&c.*

$$\left.\begin{array}{rl} y^3 = & A^3 + 3A^2 B x + 3AB^2 x^2 + B^3 x^3, \text{ \&c.} \\ & \qquad\qquad + 3A^2 C x^2 + 6ABC x^3 \\ & \qquad\qquad\qquad\qquad + 3A^2 D x^3 \\ + p^2 y = & + p^2 A + p^2 B x + p^2 C x^2 + p^2 D x^3, \text{ \&c.} \\ + p x y = & \quad \ldots \quad + p A x + p B x^2 + p C x^3, \text{ \&c.} \\ - x^3 = & \quad \ldots\ldots\ldots\ldots \quad - x^3 \\ -2p^3 = & -2p^3 \end{array}\right\} = 0.$$

Hence, (68), $A^3+p^2A-2p^3=0$ . . . . $A=p$ (57),

$3A^2B+p^2B+pA=0$ . . . $B=-\frac{1}{4}$

$3AB^2+3A^2C+p^2C+pB=0$ . . $C=\frac{1}{64p}$

$B^3+6ABC+3A^2D+p^2D+pC-1=0$. $D=-\frac{131}{512p^2}$

&c. &c.

Thus: $y=p-\frac{1}{4}x+\frac{x^2}{64p}+\frac{131x^3}{512p^2}+\frac{509x^4}{16384p^3}$, &c. as before. And this ſeries converges ſo much the faſter, the leſs the quantity x is in compariſon of p.

If we ſuppoſe $3n=3$, or $n=1$, we ſhall have -1, -2, -3, -4, -5, &c. for r, s, t, &c. And hence, the deſcending ſeries, $y=Ax+B+Cx^{-1}+Dx^{-2}$, &c. $=x-\frac{1}{3}p-\frac{p^2}{3x}+\frac{55p^3}{81x^2}+\frac{64p^4}{243x^3}$, &c. viz. by proceeding as above. Which ſeries, as is obvious, will converge ſo much the faſter, the greater x is in compariſon of p.

Ex. 3. $y^3+y^2+y-x^3=0$,

$A^3x^{3n}+A^2x^{2n}+Ax^n-x^3=0$.

Suppoſing $3n=3$, or $n=1$.

Then -1, -2, -3, &c. $=r$, s, t, &c. And,

Hence, $y=Ax+B+Cx^{-1}+Dx^{-2}$, &c.

$$\left.\begin{array}{rlllll}
y^3= & A^3x^3 & +3A^2Bx^2 & +3A^2Cx & \left.\begin{array}{r}+3A^2D\\+6ABC\\+B^3\end{array}\right\} & \text{\&c.}\\
 & & & +3AB^2x & & \\
+y^2= & & +A^2x^2 & +2ABx & \left.\begin{array}{r}+B^2\\+2AC\end{array}\right\} & \text{\&c.}\\
+y= & \ldots & \ldots & +Ax & +B & \\
-x^3= & -x^3 & & & &
\end{array}\right\}=0.$$

Then (68), $A^3 - 1 = 0$ . . . . $A = 1$
$3A^2B + A^2 = 0$ . . . . $B = -\frac{1}{3}$
$3A^2C + 3AB^2 + 2AB + A = 0$ . . . $C = -\frac{2}{9}$
$3A^2D + 6ABC + B^3 + B^2 + 2AC + B = 0$. $D = \frac{7}{81}$
&c. &c.

Thus: $y = x - \frac{1}{3} - \frac{2}{9x} + \frac{7}{81x^2}$ &c. And this ſeries, as is obvious, will converge ſo much the faſter, the greater we ſuppoſe x to be taken

If we ſuppoſe $n = 3$, then $y = Ax^3 + Bx^6 + Cx^9$, &c. for an aſcending ſeries; alſo, if we ſuppoſe $2n = n$, or $n = 0$, then $y = A + Bx^3 + Cx^6$, &c. which will give an aſcending ſeries alſo.

82. Corol. 2. Having ſubſtituted the value of n into all the exponents, as in the laſt corollary, let each of the exponents be ſubtracted from the next greater for a ſeries of *differences*, then, if the leaſt difference be divided by the *greateſt common meaſure* of all the differences, the *leaſt* number which can be divided by the quotient (40), will be ſuch a diviſor of the leaſt difference, as will give the *common* difference of all the exponents in the ſeries for y. Suppoſing, therefore, this common difference to be repreſented by r, we ſhall have the following ſeries for y, *viz.*

$$y = Ax^n + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r}, \text{ \&c.}$$

Where the common difference r muſt be taken as affirmative or negative, according as the ſeries is to aſcend or deſcend (81.)

Ex. 1. $y^3 - p^2y + pxy - x^3 - 2p^3x^0 = 0$,
$3n$, n, $n+1$, 3, 0. Suppoſe $n = 0$.

Exponents, 0, 0, 1, 3, 0.

Differences, 1, 2, whereof the greateſt common meaſure is, 1; $\frac{1}{1} = 1$, and $r = 1$, (81.)

Hence, $y = A + Bx + Cx^2 + Dx^3$, &c. as before.

If we ſuppoſe $3n = 3$, or $n = 1$.

Exponents 3, 1, 2, 3, 0.

Differences, 1, 1. Greateſt common meaſure, 1.

Hence, $r = -1$, (81.) and
$y = Ax + B + Cx^{-1} + Dx^{-2}$, &c. as before.

Ex. 2. $y^5 - p y^2 + 9 p x^2 - x^3 = 0$

$5n$, $2n$, 2, 3. Suppoſe $2n = 2$, or $n = 1$.

Exponents, 5, 2, 2, 3.

Differences, 1, 2. Greateſt common meaſure 1.

Hence, $r = 1$, and

$$y = A x + B x^2 + C x^3 + D x^4, \ \&c.$$

If we ſuppoſe $5n = 3$, or $n = \frac{3}{5}$,

Exponents, 3, $1\frac{1}{5}$, 2, 3.

Differences, $\frac{4}{5}$, 1. Greateſt common meaſure, 1.

The leaſt number, diviſible by $\frac{4}{5}$, is 4, (40.) And $\frac{4}{5}$ divided by 4, gives $\frac{1}{5}$ for the quotient

Hence, $r = -\frac{1}{5}$, and

$$y = A x^{\frac{3}{5}} + B x^{\frac{2}{5}} + C x^{\frac{1}{5}} + D, \ \&c.$$

Ex. 3. $y^3 - p x y + x^3 = 0$,

$3n$, $n + 1$, 3. Suppoſe $n + 1 = 3$, or $n = 2$.

Exponents, 6, 3, 3.

Differences, 3. Greateſt common meaſure 3.

$\frac{3}{3} = 1$, and $\frac{3}{1} = 3 = r$.

Hence, $y = A x^2 + B x^5 + C x^8$, &c.

If we ſuppoſe $3n = 3$, or $n = 1$.

Exponents, 3, 2, 3.

Differences, 1. Greateſt common meaſure, 1.

$\frac{1}{1} = 1$, and $r = -1$.

Hence, $y = A x + B + C x^{-1}$, &c.

83. Corol. 3. If it appears that A hath various values, ſome of which are equal to one another, it is obvious, that the *number* of equal values will vary according to the particular value we take of A. Thus: If a, a, a, b, b, c, repreſent ſix values of A, the number of equal values of A will be 3, 2, or 1, according as we take A equal to a, b, or c. Therefore having, as before determined, the value of n, let the differences of the exponent by which n was determined, and the other exponents, be *now* taken

as a ſeries of differences; let the leaſt difference be divided by the greateſt common meaſure of all the differences, and the leaſt number, (45), that can be divided by this quotient, and the number of equal values of A, will be, in *every caſe*, the diviſor of the leaſt difference, and the quotient the *true* value of *r*.

Ex. 1. $x^5 - 4y^{\frac{1}{2}}x^4 + 6yx^3 - 4y^{\frac{3}{2}}x^2 + y^2x + 10x^{\frac{1}{3}} - 6y^{\frac{1}{2}} = 0,$

$5,\ \frac{1}{2}n + 4,\ n + 3,\ \frac{3}{2}n + 2,\ 2n + 1,\ \frac{1}{3},\ \frac{1}{2}n.$

Suppoſing $2n + 1 = 5$, or $n = 2$.

Exponents, $5,\ 5,\ 5,\ 5,\ 5,\ \frac{1}{3},\ 1.$

Differences, $4\frac{2}{3},\ 4.$ Greateſt common meaſure, $\frac{2}{3}$.

$4 \div \frac{2}{3} = 4 \times \frac{3}{2} = 6.$

In this example, A has 4 equal values. And the leaſt number, diviſible by 6, 4, is 12, (45.)

$\frac{4}{12} = \frac{1}{3}$, and therefore, $r = -\frac{1}{3}$.

Hence, $y = Ax^2 + Bx^{\frac{5}{3}} + Cx^{\frac{4}{3}} + Dx + Ex^{\frac{2}{3}}$, &c.

Ex. 2. $-\frac{y^9}{a^5} + xy^3 - 2x^2y^2 + x^3y + \frac{x^{14}}{b^{10}} = 0,$

$9n,\ 3n + 1,\ 2n + 2,\ n + 3,\ 14.$

Suppoſing $2n + 2 = n + 3$, or $n = 1$.

Exponents, $9,\ 4,\ 4,\ 4,\ 14.$

Differences, 5, 10. Greateſt common meaſure, 5.

$5 \div 5 = 1.$

In this example, A has 2 equal values. And the leaſt number, diviſible by 1, 2, is 2, (45.)

$\frac{5}{2} = 2\frac{1}{2}$, or $r = 2\frac{1}{2}$.

Hence, $y = Ax + Bx^{3\frac{1}{2}} + Cx^6 + Dx^{8\frac{1}{2}}$, &c.

Ex. 3. $a^4y^2 - 2a^4xy + a^4x^2 + x^4y^2 = 0,$

$2n,\ n + 1,\ 2,\ 2n + 4.$ Suppoſe $2n = 2$, or $n = 1$.

Exponents, $2,\ 2,\ 2,\ 6.$

Differences, 4. Greateſt common meaſure, 4.

$\frac{4}{4} = 1.$ And A has 2 equal values.

The leaſt number, diviſible by 1, 2, is 2, (45.)

$\frac{4}{2} = 2 = r.$

Hence, $y = Ax + Bx^3 + Cx^5 + Dx^7$, &c.*

84. Corol. 4. General forms, for approximating infinite converging ſeries, may eaſily be derived from this propoſition, and *Sir Iſaac Newton's* Binomial Theorem, (78.) Thus: let

$$ax^m + bx^{m+n} + cx^{m+2n} + dx^{m+3n}, \text{ \&c.}$$

repreſent any infinite converging ſeries, and let the following general expreſſions be aſſumed as approximations to the value of the propoſed ſeries, *viz.*

$\dfrac{ax^m}{1 + Px^n}$, First approximation.

$\dfrac{ax^m + Ax^{m+n}}{1 + Px^n}$, Second approximation.

$\dfrac{ax^m + Ax^{m+n}}{1 + Px_n + Qx_{2n}}$, Third approximation.

Then by ſubſtitution, and comparing the homologous terms, (68), we have

$x^m \times \dfrac{\frac{a^2}{1 - bx^n}}{a}$, Firſt approximation, where $P = -\dfrac{b}{a}$.

$x_m \times \dfrac{ab + \overline{b^2 - ac} \times x^n}{b - cx^n}$, Second approximation, where $P = -\dfrac{c}{b}$, and $A = b - \dfrac{ac}{b}$.

$x^m \times \dfrac{a + \overline{b + aP} \times x^n}{1 + Px^n + Qx^{2n}}$, Third approximation, where $P = \dfrac{bc - ad}{ac - b^2}$, $Q = \dfrac{bd - c^2}{ac - b^2}$, and $A = \dfrac{b \times \overline{ac - b^2} + a \times \overline{bc - ad}}{ac - b^2}$ †.

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* See Sir Iſaac Newton's Analyſis of Equations, conſiſting of an infinite number of terms, with Mr Colſon's Commentary on Sir Iſaac Newton's Fluxions, s'Graveſande's Algebra, Stirling on Series, Maclaurin's Algebra, and Emerſon's Algebra.

† See Mr Simpſon's Diſſertations, p. 99.

We may alſo approximate any infinite converging ſeries, by aſſuming a fraction, whereof one of the terms is the firſt term of the propoſed ſeries, and the other conſiſts of quantities to be determined by ſuppoſing the fraction equal to two or more of the initial terms of the ſeries.

Ex. 1. Suppoſe $2ax - \frac{5ab}{2}x^2 + \overline{3ab^2 - \frac{8a^3}{3}} \times x^3$, &c. to be an infinite converging ſeries, the quantity x being ſmall.

Aſſume $\left(\frac{2ax}{A-B} =\right) 2ax - \frac{5ab}{2} = \frac{2ax}{A} + \frac{2axB}{A^2}$ . . . . ; then comparing the homologous terms, (68), we have $A = 1$, and $B = -\frac{5bx}{2}$. Thus: $\frac{2ax}{1 + \frac{5bx}{2}}$ is the firſt approximating fraction. By diviſion, $\frac{2ax}{1 + \frac{5bx}{2}} = 2ax - \frac{5ab}{2}x^2 + \frac{25ab^2}{8}x^3 - \frac{125ab^3}{32}x^4$, &c. therefore, ſuppoſing the propoſed ſeries to be repreſented by M, by ſubtraction, we have $\frac{2ax}{1 + \frac{5bx}{4}} - \frac{8a^2}{3}x_3 + \frac{125ab^3}{32}x^4$, &c. $=$ M, or,

$$\frac{2ax}{1 + \frac{5bx}{4}} - \frac{1}{3} \times \overline{\overline{2ax}^3 - 6ax \times \left.\frac{5bx}{4}\right|^3} \text{ \&c.} = M.$$

Hence, to determine a ſecond approximating fraction,

Aſſume $\left(\left.\frac{2ax}{C+D}\right|^3 =\right) \overline{2ax}^3 - 6ax \times \left.\frac{5bx}{4}\right|^3 = \frac{\overline{2ax}^3}{C^3} - \frac{3D \times \overline{2ax}^3}{C^4}$ . . . . ; then, comparing the homologous terms, we have $C = 1$, and $3D \times \overline{2ax}^3 = 6ax \times \left.\frac{5bx}{4}\right|^3$ which gives the ratio $D : 2ax = \left.\frac{5bx}{4}\right|^3 : \overline{2ax}^3$, and of conſequence, $D=\frac{5bx}{4}$. Thus:

$$\frac{2ax}{1+\frac{5bx}{4}}-\frac{1}{3}\times\left|\frac{2ax}{1+\frac{5bx}{4}}\right|^3=\text{M nearly*.}$$

Ex. 2. Suppoſe $AmL+\frac{1}{2}Am^2L^2+\frac{1}{6}Am^3L^3+\frac{1}{24}Am^4L^4$, *&c.* to be an infinite converging ſeries, $(=P.)$

Aſſume $\left(\frac{AmL}{C+D}=\right)$ $AmL+\frac{1}{2}Am^2L^2=\frac{AmL}{C}-\frac{AmLD}{C^2}$., ... ; then $C=1$, $D=-\frac{1}{2}mL$, and $\frac{AmL}{1-\frac{1}{2}mL}$ is the firſt approximating fraction. By diviſion,

$$\frac{AmL}{1-\frac{1}{2}mL}=AmL+\frac{1}{2}Am^2L^2+\frac{1}{4}Am^3L^3+\frac{1}{8}Am^4L^4+\frac{1}{16}Am^5L^5+,\ \&c.$$

therefore, by ſubtraction,

$$P=\frac{AmL}{1-\frac{1}{2}mL}-\frac{1}{12}Am^3L^3-\frac{1}{12}Am^4L^4-\frac{13}{240}Am^5L^5,\ \&c.$$

Then, for a ſecond approximating fraction,

Aſſume $\left(\frac{\frac{1}{12}Am^3L^3}{E+F}=\right)$ $\frac{1}{12}Am^3L^3+\frac{1}{12}Am^4L^4=\frac{\frac{1}{12}Am^3L^3}{E}-\frac{\frac{1}{12}Am^3L^3\times F}{E^2}$ . . . . ; then $E=1$, $F=-mL$, and therefore,

$$P=\frac{AmL}{1-\frac{1}{4}mL}-\frac{\frac{1}{12}Am^3L^3}{1-mL},\ \&c.$$

If now we reſolve the fraction $\frac{\frac{1}{12}Am^3L^3}{1-mL}$, by diviſion, and proceed as before, we ſhall obtain a third approximating fraction, *viz.* $\frac{\frac{7}{240}Am^5L^5}{1-\frac{11}{6}mL} = \frac{\frac{1}{35}Am^5L^5}{1-2mL}$ nearly. And thus:

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* This approximation is that given by Mr Simpſon, without any inveſtigation, in his ESSAYS, p. 47. for determining the equation of a planetary orbit. Very ingenious ſolutions of Kepler's Problem, have alſo been given by Sir Iſaac Newton, Mr Machin, Dr Keil, Dr Matthew Stewart, *&c.*

$$P = \frac{AmL}{1-\frac{1}{2}mL} - \frac{\frac{1}{12}Am^3L^3}{1-mL} + \frac{\frac{1}{35}Am^5L^5}{1-2mL}, \text{\&c. very nearly.}$$

Proposition XXVI.

85. Suppoſing n to repreſent a number: The meaſures of the ratios $\frac{1}{1-n}:1$, and $1+n:1$; or, the logarithms of the quantities $\frac{1}{1-n}$, and $1+n$, will be expreſſed by the following infinite series, *viz.*

$$\text{L}, \frac{1}{1-n} = M \times \overline{n + \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 + \tfrac{1}{4}n^4 + \tfrac{1}{5}n^5, \text{\&c.}}$$

$$\text{L}, 1+n = M \times \overline{n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \tfrac{1}{5}n^5, \text{\&c.}}$$

Where M, (which may be expounded by any number taken at pleaſure) is that Quantity, called by Mr Cotes, The Modulus of the Logarithmic System*.

86. Corol. 1. If we combine together the ſeries for the logarithm of $\frac{1}{1-n}$, and that for the logarithm of $1+n$, we ſhall obtain a ſingle ſeries, which will converge much faſter than either ſeries in this propoſition, *viz.*

$$\text{L}, \frac{1+n}{1-n} = 2M \times \overline{n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \tfrac{1}{7}n^7, \text{\&c.}}$$

From this ſeries, therefore, to derive the logarithm of any quantity, P, we muſt ſuppoſe $P = \frac{1+n}{1-n}$, and hence $n = \frac{P-1}{P+1}$ a proper fraction, and of conſequence, in every caſe, a converging quantity.

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* Harmonia Menſurarum, Prop. I.

87. Corol. 2. Since the *modulus* M may be interpreted by any numbers taken at pleaſure; therefore, DIFFERENT logarithmic numbers may be derived from the ſame ſeries, (85, 86), by taking for M different numbers. Thus: If the *modulus* $M = 1$, the logarithms produced will be thoſe publiſhed by the illuſtrious inventor of logarithms, *viz.* The Honourable John Napier Baron of Merchiſton; the logarithm of any quantity P, being, in this caſe, expounded by

$$2 \times \overline{n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 +}, \text{ \&c.}$$

But, if the *modulus* $M = \frac{1}{2}$, or, if $2M = 1$, that is, if we halve all Napier's logarithms, another ſeries will be produced, called NATURAL LOGARITHMS, becauſe, in this caſe, the logarithm of P, is expreſſed ſimply by the ſeries,

$$n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5, \text{ \&c.}$$

And hence the natural logarithms are properly the ORIGINAL, or RADICAL LOGARITHMIC NUMBERS.

88. Corol. 3. Let Q repreſent the infinite ſeries producing *natural* logarithms, then, ſuppoſing M, m to be the *moduli* of ANY two different logarithmic ſyſtems, it is evident, that all logarithms, in THESE ſyſtems, will be proportional to the *moduli* M, m. Therefore, ſuppoſing L to repreſent a logarithm in the ſyſtem, whereof the *modulus* is M, and l a logarithm in the other ſyſtem whoſe *modulus* is m, we have the ratio $M : m = L : l$. According to Napier's ſyſtem, where the *modulus* is unity, the logarithm of 10 is the number 2.30258, &c. and according to Napier's improved ſyſtem, as publiſhed by Mr Briggs, the logarithm of 10 is unity; therefore, by ſubſtituting theſe numbers in the above analogy, we have $m = \dfrac{1}{2.30258, \text{ \&c.}} = 0.4342944$, &c. the *modulus* of the common ſyſtem of logarithms. Thus: If we divide all the logarithms in Baron Napier's ſyſtem, by his logarithm of 10, or, if we multiply them all by the reciprocal of his logarithm of 10, we ſhall have the logarithms in the common ſyſtem.

89. Corol. 4. Suppoſe $M \times \overline{n \pm \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 \pm \tfrac{1}{4}n^4, \text{ \&c.}} = L$, then, by reverting this ſeries, (79, Caſe I) we have $n = \dfrac{L}{m} \mp \dfrac{L^2}{2.M^2} + \dfrac{L^3}{2.3.M^3} \mp \dfrac{L^4}{2.3.4.M^4}$, &c. and of conſequence,

(ſuppoſing $\frac{1}{M}=m$) $1 \mp n = 1 \mp mL + \frac{m^2L^2}{2} \mp \frac{m^3L^3}{2.3} + \frac{m^4L^4}{2.3.4}$, &c. By this ſeries, we may determine the number $1 \mp n$, from the logarithm L, being given.

90. Corol. 5. As the ſeries given in the laſt corollary, converges very ſlowly, when L is large, therefore, to remedy this defect, find, by a table of logarithms, the next leſs logarithm to that given, ſetting aſide the characteriſtics, and let A be the number correſponding thereto: Then, if z repreſent the number required, we ſhall have $\frac{z}{A} = 1 + mL + \frac{1}{2}m^2L^2 + \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4$, &c. (89), and therefore, $z = A + AmL + \frac{1}{2}Am^2L^2 + \frac{1}{6}Am^3L^3$, &c. $= A + \frac{AmL}{1-\frac{1}{2}mL} - \frac{\frac{1}{12}Am^3L^3}{1-mL} + \frac{\frac{1}{15}Am^5L^5}{1-2mL}$ very nearly, (84. *Ex.* 2.) Here, if we take the two firſt terms of the abbreviated ſeries, as an approximation, near enough for practice, we ſhall have $z = A + \frac{AmL}{1-\frac{1}{2}mL} = \frac{\overline{M+\frac{1}{2}L}\times A}{M-\frac{1}{2}L}$, and hence, the ratio $M - \frac{1}{2}L : M + \frac{1}{2}L = A : z$.

Ex. Suppoſe $z^{365} = 1.06$ to determine the value of z. Here $365 \times$ L. $z =$ L. 1.06 ($=$ L. 2 + L. 0.53) $= 0.025305865226477$ by Mr Sharp's table *, and therefore L. $z = 0.00006933113771$: By Sherwin's tables, the next leſſer logarithm to that of z, is 0.0000434, and Number $A = 1.0001$, and from the firſt term only of the logarithmic ſeries (86), we get L. 1.0001 ($=$ L. $\frac{10001}{10000}$) $= 0.00004342727682$ ($= 2M \times n$) $=$ L. A. Thus;

L, $z = 0.00006933113771$,
L, $A = 0.00004342727682$,

L, $\frac{z}{A} = 0.00002590386089 =$ L,
$1.0001 = A$

$AL = 0.000025906451236085$.

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* See Sherwin's tables, and Sharp's Geometry.

$M = 0.43429448190325\ (58)$

$\frac{1}{2}L = 0.00001295193042$

$M - \frac{1}{2}L = 0.43428152997283$

$$\frac{AL}{M - \frac{1}{2}L} = 0.00000596535874$$

And $z\left(= A + \frac{AL}{M - \frac{1}{2}L}\right) = 1.00015596535874$, by an exceeding eaſy operation.

91. Corol. 6. If, inſtead of taking the number correſponding to the next leſſer logarithm, as in the laſt corollary, we take that correſponding to the next greater, and call it B; then proceeding, as in the laſt corollary, we ſhall have $z = B - \frac{BmL}{1 + \frac{1}{2}mL} + \frac{\frac{1}{12}Bm^3L^3}{1 + mL} - \frac{\frac{1}{35}Bm^5L^5}{1 + 2mL}$, &c. and taking the two firſt terms, as an approximation near enough for practice, $z = B - \frac{BmL}{1 + \frac{1}{2}mL} = \frac{\overline{M - \frac{1}{2}L} \times B}{M + \frac{1}{2}L}$, and hence, the ratio $M + \frac{1}{2}L : M - \frac{1}{2}L = B : z$.

Ex. Suppoſe $z^{365} = 1.06$. Here, I take, from Sherwin's tables, the number correſponding to the logarithm 0.0000434, to ſix places, *viz.* $1.00016\ (= 7.6 \times 0.47 \times 0.28) = B$.

L, $B = 0.00006948155878$, by Sharp's table.

L, $z = 0.00006933113771$

L, $\frac{B}{z} = 0.00000015042102 = L$

$BL = 0.0000001504450873632$

$M + \frac{1}{2}L = 0.43429455711376$

$\frac{BL}{M + \frac{1}{2}L} = 0.0000003464102$,

And therefore, $z\left(= B - \frac{BL}{M + \frac{1}{2}L}\right)$

$= 1.00015596535897$ very nearly, as before.

Note. The *modulus* for the common logarithms, from Napier's, is,

0.43429448190325182765112891891 6, *&c* *.

Proposition XXVII.

93. In any ſeries of quantities a, b, c, d, e, f, g, &c. if there be taken all the orders of the differences of the terms, and if A, B, C, D, E, *&c*. repreſent the firſt terms of the ſeveral orders of differences: Then, if T repreſent the firſt term of the order of differences denoted by n, it will be $T = a - nb + n.\dfrac{n-1}{2}c - n.\dfrac{n-1}{2}.\dfrac{n-2}{3}d$, *&c*. Where the ſeveral coefficients of a, b, c, d, &c. are the UNCIÆ of a binomial, raiſed to a power denoted by n, (10, 78), and the quantity T itſelf will be affirmative or negative, according as n is an even or odd number.

Demonstration.

Series, a, b, c, d, e, f, &c.

First differences, $b-a$, $c-b$, $d-c$, $e-d$, $f-e$, &c.

Second, ——— $c-2b+a$, $d-2c+b$, $e-2d+c$, &c.

Third, ——— $d-3c+3b-a$, $e-3d+3c-b$, &c.

Fourth, ——— $e-4d+6c-4b+a$, &c.

&c. *&c*.

Thus:

$A = -a+b$

$B = a-2b+c$

$C = -a+3b-3c+d$

$D = a-4b+6c-4d+e$

$E = -a+5b-10c+10d-5e+f$,

&c. *&c*.

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* See, on the ſubject of logarithms, Dr Wallis's Algebra, chap. XII. Sir Iſaac Newton's Fluxions, Sharp's Geometry improved, Sherwin's tables, Mr Dodſon's Antilogarithmic Canon, Maſeres's Trigonometry, *&c*.

Or,

$$\begin{aligned}
-\mathrm{A} &= a - b \\
\mathrm{B} &= a - 2b + c \\
-\mathrm{C} &= a - 3b + 3c - d \\
\mathrm{D} &= a - 4b + 6c - 4d + e \\
-\mathrm{E} &= a - 5b + 10c - 10d + 5e - f, \\
&\quad \&c. \qquad\qquad \&c.
\end{aligned}$$

And hence, (10, 78.)

$$a - nb + n.\frac{n-1}{2}c - n\frac{n-1}{2}.\frac{n-2}{3}d + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}e, \ \&c. = \pm \mathrm{T},$$ according as n is even or odd.

Q. E. D.

Ex. 1. Required the firſt term of the 5th order of differences in the ſeries, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, *&c.*

Here $n = 5$, and therefore, $-\mathrm{T} = 1 - 5b + 10c - 10d + 5e - f = \frac{1}{32}$; or, $\mathrm{T} = -\frac{1}{32}$.

Ex. 2. Required the firſt of the 8th order of differences in the ſeries, 1, 3, 9, 27, 81, *&c.*

Here $n = 8$, and therefore, $\mathrm{T} = 1 - 8b + 28c - 56d$, &c. $= 256$.

Ex. 3. Required the firſt of the 4th order of differences in the ſeries, 1, 8, 27, 64, 125, *&c.*

Here $n = 4$, and therefore, $\mathrm{T} = a - 4b + 6c$, &c. $= 0$.

94. Corol. 1. If, from the equations in this propoſition, we derive the quantities, b, c, d, e, f, &c. in terms of a, A, B, C, D, *&c.* we ſhall have

$$\begin{aligned}
b &= a + \mathrm{A} \\
c &= a + 2\mathrm{A} + \mathrm{B} \\
d &= a + 3\mathrm{A} + 3\mathrm{B} + \mathrm{C} \\
e &= a + 4\mathrm{A} + 6\mathrm{B} + 4\mathrm{C} + \mathrm{D} \\
f &= a + 5\mathrm{A} + 10\mathrm{B} + 10\mathrm{C} + 5\mathrm{D} + \mathrm{E}, \\
&\quad \&c. \qquad\qquad \&c.
\end{aligned}$$

And hence, it appears, that the value of any term in the ſeries, a, b, c, d, e, f, &c. whoſe diſtance, from the firſt term, is denoted by n, will be $a + nA + n.\frac{n-1}{2}B + n.\frac{n-1}{2}.\frac{n-2}{3}C + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}D +$, *&c.* (10, 78.)

Or, if we take in the firſt term a, of the ſeries, a, b, c, d, e, f, &c. the nth term of this ſeries, will be

$$a + \frac{n-1}{1}A + \frac{n-1}{1}.\frac{n-2}{2}B + \frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}C + \frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}.\frac{n-4}{4}D +, \&c.$$

(10, 78.) where the coefficients are the UNCIÆ of the power denoted by $n-1$.

For
$$\begin{array}{l}
a = a \\
b = a + A \\
c = a + 2A + B \\
d = a + 3A + 3B + C \\
e = a + 4A + 6B + 4C + D \\
f = a + 5A + 10B + 10C + 5D + E, \\
\quad\quad \&c. \quad\quad\quad\quad\quad\quad \&c.
\end{array}$$

And thus it appears, that any term of a propoſed ſeries, may be *accurately* determined, when the differences of any order are equal among themſelves.

Ex. 1. Required the 20th term of the ſeries, 1, 8, 27, 64, 125, *&c.*?

Here $a=1$, $A=7$, $B=12$, $C=6$, $D=0$, and $n=20$; therefore, $a + 19A + \frac{19.18}{1.2}B + \frac{19.18.17}{1.2.3}C = 8000$, the 20th term required.

Ex. 2. Required the 20th term of the ſeries, 1, 3, 6, 10, 15, 21, *&c.*?

Here $a=1$, $A=2$, $B=1$, $C=0$, and $n=20$; therefore, $a + 19A + 19.9B = 210$.

95. Corol 2. If we take the equations in the laſt corollary, (94), expounding the terms of the propoſed ſeries, $a, b, c, d, e,$

f,

f, &c. and collect the terms into one ſum, we ſhall have the following equations, *viz.*

$$a = 0 + a$$
$$a + b = 0 + 2a + A$$
$$a + b + c = 0 + 3a + 3A + B$$
$$a + b + c + d = 0 + 4a + 6A + 4B + C$$
$$a + b + c + d + e = 0 + 5a + 10A + 10B + 5C + D$$
$$a + b + c + d + e + f = 0 + 6a + 15A + 20B + 15C + 5D + E,$$
&c. &c.

And hence (10, 78.)

$$na + n.\frac{n-1}{2}A + n.\frac{n-1}{2}.\frac{n-2}{3}B + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}C + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}D +, \&c.$$

will be the ſum of *n* terms of the ſeries, *a*, *b*, *c*, *d*, *e*, *f*, &c. By this theorem, therefore, which was firſt given by the illuſtrious Sir Iſaac Newton, we can determine the ſum of any number of terms in any propoſed ſeries.

Thus: The ſum of *n* terms of the ſeries.

$$1,\ 2,\ 3,\ 4,\ \ldots\ldots\ n = \overline{n+1} \times \frac{n}{2}.$$
$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \ldots\ldots\ n^2 = \overline{n+1} \times \frac{n}{2} \times \frac{2n+1}{3}.$$
$$1^3,\ 2^3,\ 3^3,\ 4^3,\ \ldots\ldots\ n^3 = \overline{n+1} \times \frac{n}{2} \times \frac{n.\overline{n+1}}{2}$$
&c. &c.

In theſe examples, if we ſuppoſe *n* to be *greater* than *any aſſignable number*; then,

The LIMIT of the ſum of the *infinite* ſeries.

$$1,\ 2,\ 3,\ 4,\ \ldots\ldots\ n = n \times \tfrac{1}{2}n.$$
$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \ldots\ldots,\ n^2 = n^2 \times \tfrac{1}{3}n.$$
$$1^3,\ 2^3,\ 3^3,\ 4^3,\ \ldots\ldots\ n^3 = n^3 \times \tfrac{1}{4}n.$$
&c. &c.

Univerſally, $\overset{m}{1}, \overset{m}{2}, \overset{m}{3}, \overset{m}{4}, \ldots \overset{m}{n} = \overset{m}{n} \times \frac{1}{m+1} n.$

And this general example comprehends the whole ARITHMETIC OF INFINITES.

96. Corol. 3. If we ſuppoſe the terms of the ſeries, a, b, c, d, e, f, &c. to be placed, or arranged, in ſuch a way, as that they may be at an UNIT's diſtance from each other, *viz.* $a, . b, . c, . d, . e, . f,$ *&c.* Then, if n repreſent the *place* or *diſtance* of any term, x, to be *interpolated*; it will be $x = a + nA + n . \frac{n-1}{2} B + n . \frac{n-1}{2} . \frac{n-2}{3} C +$, *&c.* (94.) And, as this value of x is perfectly accurate, when x is any term of the ſeries, a, b, c, d, e, f, &c. ſo it will be *nearly true*, when x is any intermediate term.

Ex. 1. In the ſeries, $\frac{1}{5^0}, \frac{1}{5^1}, \frac{1}{5^2}, \frac{1}{5^3}, \frac{1}{5^4},$ *&c.* let that term be required, which ſtands in the middle, between $\frac{1}{5^2}$ and $\frac{1}{5^3}$.

Here $n = 2\frac{1}{2}$, and therefore $x = a + \frac{5}{2} A + \frac{15}{8} B + \frac{5}{16} C - \frac{5}{128} D = \frac{1}{5^{2\frac{1}{2}}}$.

Ex. 2. Having given the logarithmic ſigns of 1° 0′, 1° 1′, 1° 2′, and 1° 3′, to determine the logarithmic ſign of 1° 1′ 40″.

Here $(1^\circ\, 1'\, 40'' - 1^\circ = 1'\, 40'' =)\ 1\frac{2}{3} = n.$

	A	B	C
$a = 8.2418553$			
$b = 8.2490332$	71779		
$c = 8.2560943$	70611 . .	1168	
$d = 8.2630424$	69481 . .	1130 . .	38

$x = a + \frac{5}{3} A + \frac{5}{9} B - \frac{5}{81} C = 8.2537533$ very nearly.

97. Corol. 4. If we ſuppoſe the terms of the ſeries, a, b, c, d, e, f, &c. to be *anyhow* equidiſtant, and that the quantities,

in the firſt order of differences, are ſmall, it is evident, that the quantities, in the ſucceſſive orders of differences, will continually approximate to equality; and therefore, the quantities, in the laſt order of differences, will *vaniſh*, or be equal to nothing: And thus, having any number of terms given, we can, by this propoſition, determine any quantity that is wanting in the ſeries, either *accuratety* or *nearly*, by ſuppoſing the firſt term T, (93), of the correſponding order (*n*), of differences, to be equal to nothing.

1. A	$a-b$
2. B	$a-2b+c$
3. C	$a-3b+3c+d$
4. D	$a-4b+6c+4d+e$
5. E	$a-5b+10c-10d+5e-f$
6. F	$a-6b+15c-20d+15e-6f+g$
7. G	$a-7b+21c-35d+35e-21f+7g-h$,
&c.	&c.
4. T	$a-nb+n.\frac{n-1}{2}c-n.\frac{n-1}{2}.\frac{n-2}{3}d+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}e$, &c.

Ex. 1. Having given the logarithms of the numbers, 101, 102, 104, and 105, to determine the logarithm of 103.

Here, by ſuppoſing $a-4b+6c-4d+e=0$, we have $c=\frac{4\times\overline{b+d}-\overline{a+e}}{6}=2.0128372$ nearly.

$a=2.0043214$

$b=2.0086002$

$d=2.0170333$

$e=2.0211893$

$4\times\overline{b+d}=16.1025340$

$a+e=4.0255107$

$6)\ 12.0770233\ (=2.0128372.$

Ex. 2. Having given the cube roots of the numbers, 45, 46, 47, 48, and 49, to determine that of 50.

Here, by ſuppoſing $a-5b+10c-10d+5e-f=0$, we have $f=a-5b+10c-10d+5e=3.684033$.

N	Root.
45	$3.556893 = a$
46	$3.583048 = b$
47	$3.608826 = c$
48	$3.634241 = d$
49	$3.659306 = e$

$$a + 10c + 5e = 57.941683$$
$$5b + 10d = 54.257650$$
$$f = 3.684033$$
$$= \sqrt[3]{50} \text{ nearly.}$$

98. Corol. 5. If the terms of the ſeries, a, b, c, d, e, &c. be ſuppoſed to be placed at any *unequal* diſtances, in manner following, *viz.*

—x x x x x x
M. (m) N. (p) P. (q) Q. (r) R. (s) S. . . .
a b c d e . . .
y y y y y

Let $\mathrm{A} = \frac{b-a}{p}$

$$\overset{\prime}{\mathrm{A}} = \frac{c-b}{q} \quad \mathrm{B} = \frac{\overset{\prime}{\mathrm{A}} - \mathrm{A}}{p+q}$$

$$\overset{\prime\prime}{\mathrm{A}} = \frac{d-c}{r} \quad \overset{\prime}{\mathrm{B}} = \frac{\overset{\prime\prime}{\mathrm{A}} - \overset{\prime}{\mathrm{A}}}{q+r} \quad \mathrm{C} = \frac{\overset{\prime}{\mathrm{B}} - \mathrm{B}}{p+q+r}$$

$$\overset{\prime\prime\prime}{\mathrm{A}} = \frac{e-d}{s} \quad \overset{\prime\prime}{\mathrm{B}} = \frac{\mathrm{A}^{\prime\prime\prime} - \overset{\prime\prime}{\mathrm{A}}}{r+s} \quad \overset{\prime}{\mathrm{C}} = \frac{\overset{\prime\prime}{\mathrm{B}} - \overset{\prime}{\mathrm{B}}}{q+r+s} \quad \mathrm{D} = \frac{\overset{\prime}{\mathrm{C}} - \mathrm{C}}{p+q+r+s}$$

&c. &c. &c. &c.

Then, the term y, correſponding to any place x, will be

$a + \mathrm{A} \times \overline{x-m} + \mathrm{B} \times \overline{x-m} \times \overline{x-m+p} + \mathrm{C} \times \overline{x-m} \times \overline{x-m+p} \times \overline{x-m+p+q} +$, &c. where m repreſents the diſtance (M N) of the firſt term from any given place (M.) Or, if we ſuppoſe the point M to fall upon the point N; then,

$y = a + \mathrm{A}x + \mathrm{B}x \times \overline{x-p} + \mathrm{C}x \times \overline{x-p} \times \overline{x-p-q} + \mathrm{D}x \times \overline{x-p} \times \overline{x-p-q} \times \overline{x-p-q-r} +$, &c. continued to the ſame nnmber of terms with the ſeries a, b, c, d, e, &c.

The inveſtigation of this general theorem, for interpolating any term of a ſeries, is here omitted, as being more tedious than difficult.

Ex. 1. Given a comet's diſtance from the ſun, on the following days, at 12 at night, to find its diſtance, December 20, at the ſame hour.

December 12 . . . diſtance $301 = a$
21 . . . . $620 = b$
24 . . . . $715 = c$
26 . . . . $772 = d$.

Here $x\ (= 20 - 12) = 8$, the time after Dec. 12.
$p\ (= 21 - 12) = 9$, $q\ (= 24 - 21) = 3$, and $r\ (= 26 - 24) = 2$.
Alſo $A = \frac{319}{9} = 35.4444$. $B = -\frac{3.778}{12} = -0.314$, and $D = -\frac{0.3185}{14} = -0.0227$; whence, then $y = 586.9831$ the comet's diſtance, Dec. 20, at 12 at night.

Ex. 2. Given three diſtances of the ſun from the tropic, and the times of obſervation. To determine the time of the ſolſtice,

December 20 . . . . diſt. $39'' = a$
21 . . . . $6 = b$
24 . . . . $74 = c$

Here $p\ (= 21 - 20) = 1$, and $q\ (= 24 - 21) = 3$. Alſo, $A = -33$, and $B = \frac{55\frac{2}{3}}{4} = 13\frac{11}{12}$; whence $y = 39 - 46\frac{11}{12}x + 13\frac{11}{12}x^2 = 0$, y being nothing at the time of the ſolſtice: And therefore, $x = 1.3132$, and $x = 2.058$, (26, 27.) *.

99. Corol. 6. In the ſeries, *a*, *b*, *c*, *d*, *e*, *f*, &c. if we ſuppoſe, that there is the *ſame* ratio between every two adjacent or

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* See Sir Iſaac Newton's Principia and Quadrature of Curves,—Philoſophical Tranſactions, No 362,—Cotes's Harmonia Menſurarum,—Stirling on Series,—Simpſon's Eſſays, *&c.* and Emerſon's Conic Sections.

contiguous terms, then, are the quantities, a, b, c, d, e, f, &c. ſaid to be in *continued proportion*: And, in any ſeries of this kind, all the orders of differences are in the *ſame* continued proportion, alſo. For, ſince by ſuppoſition, the ratio $a:b=b:c=c:d=d:e=e:f$, &c. therefore, by converſion and alternation, the ratio $(a:b=)\ a-b:b-c=b-c:c-d=c-d:d-e=d-e:e-f$, &c. (13); and, by continual converſion and alternation, the ſame will appear with regard to *all* the orders of differences.

100. Corol. 7. In the ſeries of the laſt corollary, let the ratio $m:1$ be the common ratio of the ſeries; then, if we ſuppoſe this ratio to approach nearer to a ratio of equality, than to any other aſſignable ratio whatſoever, it is evident, that we ſhall have the firſt differences, $a-b$, $b-c$, $c-d$, $d-e$, $e-f$, &c. very nearly equal to one another. And therefore, in this caſe, *Any power or root of any term of the ſeries will have the ſame ratio to the difference of the ſame powers or roots of any two adjacent terms, as the term firſt aſſumed to ſuch a multiple of the common difference of the terms, as is denoted by the index of the given power or root*, NEARLY. Thus:

$$\left.\begin{array}{rl} \text{The ratio } & a^2 : a^2 - b^2 = a : 2 \times \overline{a-b} \\ & a^3 : a^3 - b^3 = a : 3 \times \overline{a-b} \\ & a^4 : a^4 - b^4 = a : 4 \times \overline{a-b} \\ \text{Univerſally, } & a^m : a^m - b^m = a : m \times \overline{a-b} \end{array}\right\}\ \text{nearly.}$$

Alſo,

$$\left.\begin{array}{rl} \text{The ratio } & a^{\frac{1}{2}} : a^{\frac{1}{2}} - b^{\frac{1}{2}} = a : \frac{1}{2} \times \overline{a-b} \\ & a^{\frac{1}{3}} : a^{\frac{1}{3}} - b^{\frac{1}{3}} = a : \frac{1}{3} \times \overline{a-b} \\ & a^{\frac{1}{4}} : a^{\frac{1}{4}} - b^{\frac{1}{4}} = a : \frac{1}{4} \times \overline{a-b} \\ \text{Univerſally, } & a^{\frac{m}{n}} : a^{\frac{m}{n}} - b^{\frac{m}{n}} = a : \frac{m}{n} \times \overline{a-b} \end{array}\right\}\ \text{nearly.}$$

And hence we derive a *very eaſy method* of approximating the roots of all ſurds or radical quantities.

Ex. 1. Suppoſe $a=2$, and $b=\sqrt{2}$. (80.)

Then $a^2-b^2 : 2 \times \overline{a-b} = a^2 : a$, which gives $b=\frac{3}{2}$ nearly.

Alſo, $a^2 - b^2 : 2 \times \overline{a - b} = b^2 : b$, which gives $b = \frac{4}{3}$ nearly. As the former approximating quantity $\frac{3}{2}$ is greater than the root, and the latter nearly as much leſs, therefore, by taking the *intermediate* quantity, we obtain $\frac{17}{12}$ a near approximation. And by repeating this operation, the root may be determined to any degree of accuracy.

Ex. 2. Suppoſe $a = 3$, and $b = \sqrt{12}$. (80.)

Here $b^2 - a^2 : 2 \times \overline{b - a} = a^2 : a$, which gives $b = \frac{7}{2}$.

Alſo $b^2 - a^2 : 2 \times \overline{b - a} = b^2 : b$, which gives $b = \frac{24}{7}$.

And by taking the *intermediate* quantity, we obtain the fraction $\frac{97}{28}$ a near approximation.

Ex. 3. Suppoſe $a = 1$, and $b = \sqrt[365]{1.06}$ (90, 91.)

Here $b^{365} - a^{365} : 365 \times \overline{b - a} = a^{365} : a$, from which $b = \frac{365.06}{365}$.

Alſo $b^{365} - a^{365} : 365 \times \overline{b - a} = b^{365} : b$, from which $b = \frac{386.9}{386.84}$.

And by taking the *intermediate* quantity, we obtain 1.0001597 a very near approximation to the root.

PROPOSITION XXVIII.

101. If we aſſume any quantity at pleaſure, and therefrom derive a ſeries of terms, increaſing or decreaſing, *by equal differences*; the terms of the ſeries are ſaid to be in CONTINUED ARITHMETICAL PROGRESSION: And one half the number of terms in the ſeries, will have the ſame *ratio* to unity, as the ſum of all the terms to the ſum of the two extreme terms.

DEMONSTRATION.

Let a, v, repreſent the two extreme terms in a continued arithmetical progreſſion, whereof d is the *common* difference of the terms, n the number, and s the ſum of all the terms: Then, ac-

according as we take a or v for the leading quantity, the ſeries will be expreſſed by the following terms, *viz.*

$$a,\ a\pm d,\ a\pm 2d,\ a\pm 3d,\ a\pm 4d\ .\ a\pm\overline{n-1}\times d\,(=v),$$

$$v,\ v\mp d,\ v\mp 2d,\ v\mp 3d,\ v\mp 4d\ .\ v\mp\overline{n-1}\times d\,(=a.)$$

And theſe ſeries being perfectly equivalent (taking the terms of the latter backwards, or from right to left), it is manifeſt, that the two taken together, will be double any one of them; and therefore, $\overline{a+v} + \overline{a+v} + \overline{a+v} + \overline{a+v}$, &c. to n terms, $=\overline{a+v}\times n$ (5) $=2s$, from whence the ratio $\frac{n}{2} : 1 = s : a+v$, (16.)

Q. E. D.

102. Corol. 1. Since, then $a\pm\overline{n-1}\times d=v$, and $v\mp\overline{n-1}\times d=a$, (101), therefore, $\overline{n-1}\times d=a\backsim v$, and hence the *ratio* $1 : n-1 = d : a\backsim v$, (16.) Thus: In any continued arithmetical progreſſion, unity hath the ſame *ratio* to the exceſs of the number of terms above unity, as the common difference of the terms to the difference of the two extreme terms.

103. Corol. 2. Hence, in a continued arithmetical progreſſion, if *any three* of the five quantities, a, v, d, n, and s, are ſuppoſed to be GIVEN, the remaining *two* may be determined, as in the following table, where the expreſſions belong to an increaſing progreſſion, and may be applied alſo to a decreaſing progreſſion, by making the quantities a and v, change places every where.

Caſes	Data	Quæſita	DETERMINATION.
I.	$a, d, n,$	v	$=a+\overline{n-1}\times d$, (101, 102)
		s	$=n\times\overline{a+d\times\frac{n-1}{2}}$

II.	a, d, v	n	$=\frac{v-a}{d}+1$
		s	$=\frac{v+a}{2}\times\overline{\frac{v-a}{d}+1}$
III.	a, d, s	n	$=\frac{-a+\sqrt{\overline{a\times\overline{a-d}+d\times\overline{2s+d}}}}{d}+\frac{1}{2}$
		v	$=\sqrt{\overline{a\times\overline{a-d}+d\times\overline{2s+d}}}-\frac{1}{2}d$
IV.	a, v, s	n	$=\frac{2s}{v+a}=2s\times\frac{1}{v+a}$ (101.)
		d	$=\frac{\overline{v+a}\times\overline{v-a}}{2s-\overline{v+a}}$
V.	a, n, s	v	$=\frac{2s}{n}-a=\overline{2s-na}\times\frac{1}{n}$ (101.)
		d	$=\frac{1}{n-1}\times\frac{2s-2na}{n}=\frac{1}{n-1}\times\overline{\frac{2s}{n}-2a}$
VI.	a, n, v	d	$=\frac{v-a}{n-1}=\overline{v-a}\times\frac{1}{n-1}$ (102.)
		s	$=\frac{n}{2}\times\overline{a+v}=n\times\frac{a+v}{2}$ (101.)

VII.	d, v, n	a	$=v-\overline{n-1}\times d$ (101.)
		s	$=n\times\overline{v-d\times\frac{n+1}{2}}$
VIII.	d, v, s	a	$=\frac{1}{2}d\pm\sqrt{v\times\overline{v+d}+d\times\overline{2s-\frac{1}{4}d}}$
		n	$=\frac{v\mp\sqrt{v\times\overline{v+d}+d\times\overline{2s-\frac{1}{4}d}}}{d}+\frac{1}{2}$
IX.	d, n, s	a	$=\frac{s}{n}-\frac{1}{2}d\times\overline{n-1}$ (101, 102.)
		v	$=\frac{s}{n}+\frac{1}{2}d\times\overline{n-1}$ (101, 102.)
X.	n, v, s	a	$=\frac{2s}{n}-v=\overline{2s-nv}\times\frac{1}{n}$ (101.)
		d	$=\frac{2v}{n}+\frac{2}{n}\times\frac{v-s}{n-1}.$

104. Corol. 3. Since univerſally $s=\frac{n}{2}\times\overline{a+v}$ (101); therefore, if $a=0$, then $s=\frac{n}{2}\times v$. Thus: the ſum of a continued arithmetical progreſſion, whereof one of the extreme terms is a cypher, (0), is equal to the other extreme term repeated as often as there are units in half the number of terms in the ſeries.

105. Corol. 4. From Caſe I. in Corol. 2. it appears that $s=n\times\overline{a+d\times\frac{n-1}{2}}$; therefore, if we ſuppoſe $a=2$, and $d=2$ alſo, then $s=n^2+n$: And, in this caſe, s is called a Pro-

NIC NUMBER. Thus: *pronic* numbers are produced by adding together *even* numbers in a continued arithmetical progreſſion.

Continued Arithmetical Progreſſion, 2, 4, 6, 8, 10, 12, 14, 16, &c.
Pronic Numbers Numbers, 2, 6, 12, 20, 30, 42, 56, 72, &c.

Moreover, ſince $n^2 + n = s$, therefore, (26), $n = \frac{\sqrt{4s+1} - 1}{2}$ the pronic root of the number denoted by s: And hence, *if we increaſe the quadruple of any propoſed number by unity; then, half the exceſs of the ſquare root of this ſum, above unity, will be the pronic root of the propoſed number.*

106. Corol. 5. Further, ſince univerſally $s = n \times \overline{a + d \times \frac{n-1}{2}}$; therefore, if we ſuppoſe $a = 1$, and $d = 2$, then $s = n^2$: And, thus all ſquare numbers may be conſidered as originating from the addition of odd numbers in a continued arithmetical progreſſion.

Continued Arithmetical Progreſſion, 1, 3, 5, 7, 9, 11, 13, 15, 17, &c.
Square Numbers, 1, 4, 9, 16, 25, 36, 49, 64, 81, &c.

107. Corol. 6. From the laſt corollary may be derived a general theorem, for determining *how* many odd numbers muſt be added together to produce a *given power of that number*. Let the given number be n, the exponent of its power m, and the firſt term of the ſeries a; then, ſince in a ſeries of odd numbers, the difference of the terms is 2, (106), and number of terms n, by ſuppoſition; therefore, $(s =)\, n^m = n \times \overline{a + \overline{n - 1}}$, and of conſequence $a = n^{m-1} - \overline{n - 1}$: And thus it appears, how powers of all dimenſions may be obtained by the SUMMING up of *odd* numbers.

Thus: Suppoſing $m = 3$, and expounding n by the numbers 2, 3, 4, 5, 6, 7, &c.

$2^3 = 3 + 5$
$3^3 = 7 + 9 + 11$
$4^3 = 13 + 15 + 17 + 19$
$5^3 = 21 + 23 + 25 + 27 + 19$
$6^3 = 31 + 33 + 35 + 37 + 39 + 41$,
&c. &c.

Again: Suppoſing $m = 4$, and expounding n as before.

$2^4 = 7 + 9$
$3^4 = 25 + 27 + 29$
$4^4 = 61 + 63 + 65 + 67$
$5^4 = 121 + 123 + 125 + 127 + 129$
$6^4 = 211 + 213 + 215 + 217 + 219 + 221,$
&c. &c.

And we may proceed in the ſame manner for any other powers.

108. Corol. 7. If we take the common arithmetical progreſſion of natural numbers, proceeding regularly from unity, and terminating in any *ſquare number*, the terms of this progreſſion may be ſo diſpoſed within the cells of a geometrical ſquare, that the ſums of each row, taken diagonally, laterally and vertically, ſhall be the ſame; and ſuch arithmetical ſquares are called MAGIC SQUARES. Here, as the ſide (or root) of the terminating ſquare number, may be either odd, evenly-even, or unevenly-even, ſo, in the conſtruction of magic ſquares, there may be three different caſes.

109. Caſe 1. When the ſide of the ſquare number is odd. Range all the numbers in their natural order within the cells of a geometrical ſquare; this ſquare is called *the generator*, or *generating ſquare*. The diagonal and the lateral middle column of the generator, give the vertical middle column and the diagonal of the magic ſquare; at the extremities of the diagonal of the magic ſquare, deſcend and aſcend vertically, one term, increaſing and decreaſing; continue upon diagonals, deſcending and aſcending vertically at their extremities as before; and the remaining cells will be filled up with terms contiguous to thoſe, at oppoſite extremeties of adjoining columns vertically and laterally, continuing upon diagonals as before.

GENERATOR.
Square of 3.

1	2	3
4	5	6
7	8	9

MAGIC SQUARE.
Side 3.

8	1	6
3	5	7
4	9	2

GENERATOR.
Square of 5.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

MAGIC SQUARE.
Side 5.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

MAGIC SQUARE.
Side 7.

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	24	33	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

MAGIC SQUARE.
Side 9.

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

110. Caſe 2. When the ſide of the ſquare number is evenly-even, or diviſible by 4 †.

Here we muſt firſt conſtruct the magic ſquare, whoſe ſide is 4, the leaſt evenly-even number. In order to which, having formed the generator as in caſe 1, let the terms on both diagonals remain, and exchange the other exterior terms croſs-ways with thoſe that are oppoſite to them, and the ſquare will be completed.

GENERATOR.
Square of 4.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

MAGIC SQUARE.
Side 4.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

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† See Memoires de l'Academie Royale des Sciences pour l'année 1750.

Having thus obtained the magic ſquare of 4, all other magic ſquares, whoſe ſides are evenly-even, may be derived by ranging all the terms of the progreſſion into two rows; the terms in the upper row called *ſmall* numbers, being in the natural order from unity; and the terms of the lower row called the *complements* of the ſmall numbers, being in a retrograde order, ſeparate both rows, by drawing a line through them at every eight number; then theſe parcels of ſixteen figures muſt be formed into ſquares, according to the method uſed above in conſtructing the magic ſquare of 4, and the magic ſquare will be completed, by joining theſe ſubſidiary ſquares, ſo as to form a ſquare any how.

The Conſtruction of the MAGIC SQUARE, *whoſe ſide is* 8.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
64	63	62	61	60	59	58	57	56	55	54	53	52	51	50	49	48	47	46	45	44	43	42	41	40	39	38	37	36	35	34	33

GENERATORS.

1	2	3	4
5	6	7	8
57	58	59	60
61	62	63	64

9	10	11	12
13	14	15	16
49	50	51	52
53	54	55	56

17	18	19	20
21	22	23	24
41	42	43	44
45	46	47	48

25	26	27	28
29	30	31	32
33	34	35	36
37	38	39	40

SUBSIDIARY SQUARES.

1	63	62	4
60	6	7	57
8	58	59	5
61	3	2	64

9	55	54	12
52	14	15	49
16	50	51	13
53	11	10	56

17	47	46	20
44	22	23	41
24	42	43	21
45	19	18	48

25	39	38	28
36	30	31	33
32	34	35	29
37	72	26	40

MAGIC SQUARE.
Side 8.

1	63	62	4	9	55	54	12
60	6	7	57	52	14	15	49
8	58	59	5	16	50	51	13
61	3	2	64	53	11	10	56
17	47	40	20	25	39	38	28
44	22	23	41	36	30	31	33
24	42	43	21	32	34	35	29
45	19	18	48	37	27	26	40

MAGIC SQUARE.

Side 12.

1	143	142	4	9	135	134	12	17	127	126	20
140	6	7	37	132	14	15	129	124	22	23	121
8	138	139	5	16	130	131	13	24	122	123	21
141	3	2	144	133	11	0	136	125	19	18	128
25	119	118	28	33	111	110	36	41	105	102	44
116	30	31	113	108	38	39	105	100	46	47	97
32	114	115	29	40	106	107	37	48	96	99	45
117	27	26	120	109	35	34	112	101	43	42	104
49	95	94	52	57	87	86	60	65	79	78	68
92	54	55	89	84	62	63	81	76	70	71	73
56	90	91	53	64	82	83	61	72	74	75	79
93	51	50	96	85	59	58	88	77	67	66	80

111. Caſe 3. When the ſide of the ſquare number is unevenly-even.

Here we muſt firſt conſtruct the magic ſquare whoſe ſide is 6, the ſmalleſt unevenly-even number. In order to which, let all the numbers in the regular progreſſion be diſtributed into two rows, as in the laſt caſe, in manner following:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,
36, 35, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 20, 19,

Then the ſquare of 4 being the greateſt evenly-even ſquare that can be inſcribed in the ſquare of 6, we may take the eight laſt of the ſmall numbers, with their complements for filling up this ſquare, and the ten preceding numbers, with their complements, remain to fill up the twenty cells of the *border*; or, we may take the firſt eight numbers with their complements, or thoſe in the middle to fill the ſquare, while the other extreme numbers, or thoſe in the middle and extremes remain for the border. Here then we ſhall take the firſt ten ſmall numbers, with their complements for the border, *viz.*

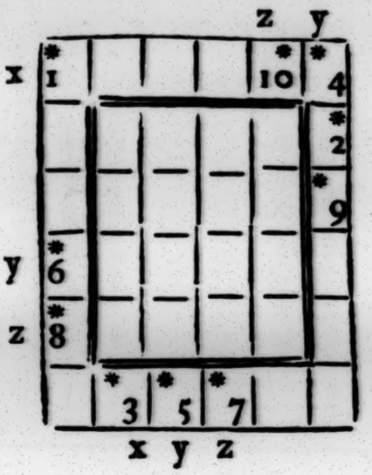
1, 2, 3, 4, 5, 6, 7, 8, 9, 10,
36, 35, 34, 33, 32, 31, 30, 29, 28, 27,

$\overset{\times}{x}$				$\overset{\times}{z}$	$\overset{\times}{y}$
	11	25	24	14	$\overset{\times}{x}$
	22	16	17	12	$\overset{\times}{z}$
$\overset{\times}{y}$	18	20	21	16	
$\overset{\times}{z}$	23	13	12	26	
	$\overset{\times}{x}$	$\overset{\times}{y}$	$\overset{\times}{z}$		

The internal part of the ſquare being conſtructed, as in the laſt caſe (110), and each ſmall number with its complement, making up the ſelf-ſame ſum, therefore any ſmall number that is placed in the border, muſt have its complement oppoſite to it; and any ſmall number, placed at one extremity of either diagonal, muſt have its complement at the other extremity of the ſame diagonal. If now we mark ten cells in the border, each with an aſteriſm (*) for the ten ſmall numbers, then the three ſmall numbers to be placed in each row of the border, making up the ſame ſum, may be expreſſed in general terms, by x, y, and z, and 1, 2, 3, &c. to 10 terms, being equal to 55, (101) therefore, ($3x+3y+4z$ or) $3\times\overline{x+y}+4z=55$ which indeterminate equation being reſolved by Prop. XIX. and its corrollaries (40—48) we obtain four different general methods for placing the ſmall numbers in the proper cells of the border, *viz.*

$$\begin{array}{rl|l} x+y= & 17,\ 13,\ 9,\ 5 & 1 \\ z= & 1,\ 4,\ 7,\ 10 & 13 \\ \hline x+y+z= & 18,\ 17,\ 16,\ 15 & \end{array}$$

and as the ſmall numbers x, y, z, to be placed in the border, may be derived from any one of theſe equations, we ſhall therefore take the laſt values of $x+y$ and z, theſe being in the *leaſt* numbers. Wherefore, having placed 10, the value of z, in the cell of the upper row belonging to it, ſince the two ſmall numbers x, y, to be placed in the extreme cells of this row taken together, amount to 5, we may therefore take either the numbers 1 and 4,



or 2 and 3; let us take then the numbers 1 and 4, and expunge theſe with the number 10, out of the ſeries of ſmall numbers

~~1~~, 2, 3, ~~4~~, 5, 6, 7, 8, 9, ~~10~~.

Since the two ſmall numbers x, z, to be placed in the right hand row of the border, taken together, amount to 11, we may take for them either 2 and 9, or 3 and 8; let us take then the numbers 2 and 9, and expunge them out of the ſeries of ſmall numbers.

~~1~~, ~~2~~, 3, ~~4~~, 5, 6, 7, 8, ~~9~~, ~~10~~,

And ſince the ſmall numbers y, z, to be placed in the left hand row of the border, amount to 14, we have therefore 6 and 8 to be placed in theſe cells; wherefore, expunging theſe numbers out of the ſeries of ſmall numbers

~~1~~, ~~2~~, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~,

we have remaining for the three cells of the lower border, the ſmall numbers 3, 5, 7. The border is then completed, by writing oppoſite to each ſmall number, x, y, z, its reſpective complement.

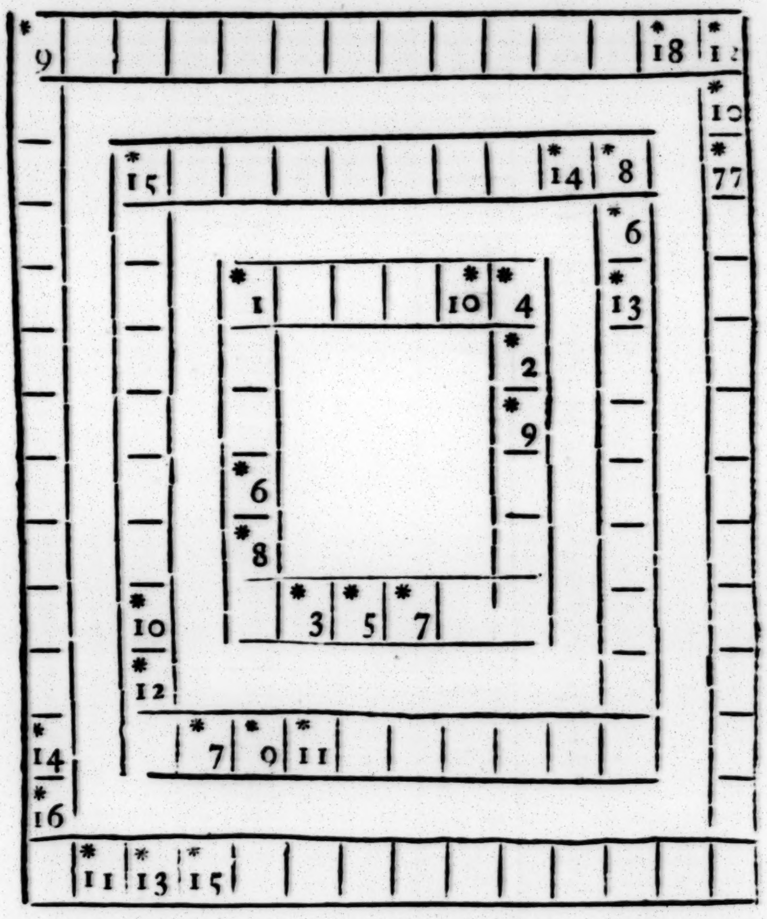
MAGIC SQUARE.
Side 6.

*1	34	32	30	*10	*4
35	11	25	24	14	*2
28	22	16	17	19	*9
*6	18	20	21	16	31
*8	23	13	12	6	29
33	*3	*5	*7	27	36

112.

112. Having thus obtained the magic ſquare, whoſe ſide is 6, we may, in the ſelf-ſame manner, conſtruct the magic ſquare of any other number, whoſe ſquare root is unevenly-even. Alſo, in conſtructing magic ſquares, whoſe ſides are numbers unevenly-even, we may proceed in this manner, *viz.*

Diſtribute all the numbers of the progreſſion into two rows, as in the laſt caſe, ſeparating them into parcels of ſixteen figures, by drawing a line at every eight number, reckoning from the furthermoſt extremity; complete theſe ſquares, and join them into one, as in the laſt caſe; and thus we ſhall have the internal part of the ſquare. To obtain the numbers to be placed in the border, from the number denoting the ſide of the ſquare to be conſtructed, take the number 6, and add the remainder to the numbers in the extreme cells of the ſquare of 6, marked with an aſteriſm (*) placing the ſums in the correſponding cells of the ſquare to be conſtructed, and their



complements in the oppoſite extreme cells. Laſtly, take the remaining ſmall numbers, and diſtribute them into two rows, placing the greater under the leſſer as before; ſeparate the numbers in theſe two rows into four diviſions, and place all the numbers of the ſame diviſion in the ſame row of the border, which will be completed by writing the complements of the ſmall numbers in the cells oppoſite to them.

The Conſtruction of the MAGIC SQUARE, *whoſe ſide is* 10.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
100	99	98	97	96	95	94	93	92	91	90	89	88	87	86	85	84	83	82	81	80	79	78	77	76

26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
75	74	73	72	71	70	69	68	67	66	65	64	63	62	61	60	59	58	57	56	55	54	53	52	51

Small numbers produced by adding 4 (=10—6) to the numbers in the aſteriſm cells of the ſquare of 6.

5, 6, 7, 8, 9, 10, 11, 12, 13, 14

Remaining ſmall numbers diſpoſed according to the rule, and ſeparated into four diviſions.

1	2	3	4
18	17	16	15

MAGIC SQUARE.

Side 10.

*5	94	92	90	85	98	1	18	*14	*8
95	19	81	80	22	27	73	72	30	*6
88	78	24	25	75	70	32	33	67	*13
99	26	76	77	23	34	68	69	31	2
84	79	21	20	82	71	29	28	74	17
15	35	65	64	38	43	57	56	46	86
4	62	40	41	59	54	48	49	51	97
*10	42	60	61	39	50	52	53	47	91
*12	63	37	36	66	55	45	44	58	89
93	*7	*9	*11	16	3	100	83	87	96

The Conſtruction of the MAGIC SQUARE, *whoſe ſide is* 14.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
196	195	194	193	192	191	190	189	188	187	186	185	184	183	182	181	180	179	178	177	176	175	174	173	172	171

27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
170	169	168	167	166	165	164	163	162	161	160	159	158	157	156	155	154	153	152	151	150	149	148	147

51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74
146	145	144	143	142	141	140	139	138	137	136	135	134	133	132	131	130	129	128	127	126	125	124	123

75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98
122	121	120	119	118	117	116	115	114	113	112	111	110	109	108	107	106	105	104	103	102	101	100	99

Small numbers ſeparated into four diviſions.

1		3	4	5	6	7	8
26	25	24	23	22	21	20	19

Small numbers in the aſteriſm cells.
9, 10, 11, 12, 13, 14, 15, 16, 17, 18.

Magic Square.
Side 14.

*9	186	184	182	192	175	191	176	1	26	2	25	*18	*12
187	27	169	168	30	35	161	160	38	43	153	152	46	*10
180	166	32	33	163	158	40	41	155	150	48	49	147	*17
194	34	164	165	31	42	156	157	39	50	148	149	47	3
173	167	29	28	170	159	37	36	162	151	45	44	154	24
193	51	145	144	54	59	137	136	62	67	129	128	70	4
174	142	56	57	139	134	64	65	131	126	72	73	123	23
7	58	140	141	55	66	132	133	63	74	124	125	71	190
20	143	53	52	146	135	61	60	138	127	69	68	130	177
8	75	121	120	78	83	113	112	86	91	105	104	94	189
19	118	80	81	115	110	88	89	107	102	96	97	99	178
*14	82	116	117	79	90	108	109	87	98	100	101	95	183
*16	119	77	76	122	111	85	84	114	103	93	92	106	181
185	*11	*13	*15	5	22	6	21	196	171	195	172	179	188

113 Corol. 8. The general increaſing ſeries of this propoſition a, $a + d$, $a + 2\,d$, $a + 3\,d$, $a + 4\,d$, ... $a + \overline{n - 1} \times d$ (101) becomes 1, $1 + d$, $1 + 2\,d$, $1 + 3\,d$, $1 + 4\,d$, ... $1 + \overline{n - 1} + d$ ſuppoſing a equal to unity; and combining together the terms of this latter ſeries, according to the *number of terms*, we obtain the following ſeries, expreſſing all *combinatory or polygonal numbers*, viz. 1, $\overline{1 + 1 + d}$, $1 + \overline{1 + d} + \overline{1 + 2d}$, $1 + \overline{1 + d}$, $+ \overline{1 + 2\,d}$, $+ \overline{1 + 3\,d}$, $1 + \overline{1 + d}$, $+ \overline{1 + 2\,d}$, $+ \overline{1 + 3\,d}$, $+ \overline{1 + 4\,d}$, ... $1 + \dfrac{\overline{n - 1} \times d}{2} \times n$ (103, 105).

Thus expounding the common difference *d* by 0, 1, 2, 3, 4, 5, &c. we have the following ſeries of arithmetical progreſſionals, and of polygonal numbers, *viz.*

	ARITHMETICAL PROGRESSIONS.		POLYGONAL NUMBERS.	
$d=0$	1, 1, 1, 1, 1, 1, 1, 1, 1, &c.	Units.	1, 2, 3, 4, 5, 6, 7, 8, 9, &c.	Laterals
$d=1$	1, 2, 3, 4, 5, 6, 7, 8, 9, &c.	Laterals.	1, 3, 6, 10, 15, 21, 28, 36, 45, &c.	Triangulars.
$d=2$	1, 3, 5, 7, 9, 11, 13, 15, 17, &c.		1, 4, 9, 16, 25, 36, 49, 64, 81, &c.	Quadrangulars.
$d=3$	1, 4, 7, 10, 13, 16, 19, 22, 25, &c.		1, 5, 12, 22, 35, 51, 70, 92, 117, &c.	Pentangulars.
$d=4$	1, 5, 9, 13, 17, 21, 25, 29, 33, &c.		1, 6, 15, 28, 45, 66, 91, 120, 153, &c.	Hexagonals.
$d=5$	1, 6, 11, 16, 21, 26, 31, 36, 41, &c.		1, 7, 18, 34, 55, 81, 112, 148, 209, &c.	Heptagonals.
&c.	&c.		&c.	

And as the laſtterm of the combinatory ſeries $1+\frac{\overline{\overline{n-1}\times d}}{2}\times n$ repreſents, in general, any polygonal number; therefore, expounding *d* as before, by 0, 1, 2, 3, 4, 5, &c.

$1\,.\,n\,.$ Lateral number.	$1\,.\,n\,.\,\frac{n+1}{2}$ Triangular number.
$1\,.\,n\,.\,n$ Quadrangular or ſquare number.	$1\,.\,n\,.\,\frac{3n-1}{2}$ Pentagonal number.
$1\,.\,n\,.\,\overline{2n-1}$ Hexagonalnumber.	$1\,.\,n.\,\frac{5n-3}{2}$ Heptagonal number.
&c.	

Where a point is uſed inſtead of the ſign of multiplication.

114. Taking the ſeries of units, and combining its terms continually, as alſo the terms of every ſucceeding ſeries, we obtain all the orders of *triangular figurate numbers*, Thus:

Figurates of the 1ſt order, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c. Units.
——— 2d order, 1, 2, 3, 4, 5, 6, 7, 8, 9. Laterals.
——— 3d order, 1, 3, 6, 10, 15, 21, 28, 36, 45, &c. Triangulars.
——— 4th order, 1, 4, 10, 20, 35, 56, 84, 120, 125, &c. 1ſt Triangular Pyramidals.
——— 5th order, 1, 5, 15, 35, 70, 126, 210, 330, 495, &c. 2d Triangular Pyramidals.
——— 6th order, 1, 6, 21, 56, 126, 252, 462, 792, 1287, &c. 3d Triangular Pyramidals.
&c. &c. &c.

The numbers in theſe ſeries taken vertically, being evidently the ſame with the *unciæ* of the correſponding terms of a binomial, raiſed to all integral powers (9); therefore, if we ſuppoſe n to repreſent the ſide of a figurate number of any order, or which is the ſame, the exponent of any order, then 1, $1.n$, $1.n.\frac{n+1}{2}$, $1.n.\frac{n+1}{2}.\frac{n+2}{3}$, $1.n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}$, &c. will repreſent the firſt, ſecond, third, fourth, fifth, &c. figurative numbers of that order; or ſuppoſing n as before (113.) to repreſent the number of terms, the terms of this ſeries, after the firſt, will repreſent all triangular figurate numbers, Thus:

$1.n$ Lateral number.

$1.n.\frac{n+1}{2}.\frac{n+2}{3}$ 1ſt Triangular Pyramidal number.

$1.n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}.\frac{n+4}{5}$ 3d Triangular Pyramidal number.

&c.

$1.n.\frac{n+1}{2}$ Triangular number.

$1.n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}$ 2d Triangular Pyramidal number.

&c.

The ſame expreſſions may be univerſally derived from the general polygonal number $1+\frac{\overline{n-1}\times d}{2}\times n$ by the ſecond corollary of the laſt Propoſition (95). Thus, $1+\frac{\overline{\overline{n-1}\times d}}{3}\times n\cdot\frac{n+1}{\text{[illegible]}}$ or, (taking $d=1$) $n\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}$ repreſents any 1ſt Triangular Pyramidal; and hence all the other orders of Triangular Pyramidals as above.

115. Theſe expreſſions give us alſo all the *permutations* and all the *different combinations* of quantities. The continual product of the ſeries of laterals n, $n-1$, $n-2$, $n-3$, $n-4$, &c. continued to the units in n, gives directly all the permutations of any number (n) of quantities; and if any quantity occur twice, thrice, four times, &c. then $\frac{n\cdot\overline{n-1}\cdot\overline{n-2}\cdot\overline{n-3}\cdot\overline{n-4}\cdot\&c}{2\cdot1}$, $\frac{n\cdot\overline{n-1}\cdot\overline{n-2}\cdot\overline{n-3}\cdot\overline{n-4}\cdot\&c.}{3\cdot2\cdot1}$, $\frac{n\cdot\overline{n-1}\cdot\overline{n-2}\cdot\overline{n-3}\cdot\overline{n-4}\cdot\&c}{4\cdot3\cdot2\cdot1}$, &c. will be the reſpective permutations. And, as for the combinations of quantities, theſe, when the quantities are taken by two's, three's, four's, &c. go on according to the numbers in the ſeries of triangulars, 1ſt triangular pyramidals, 2d triangular pyramidals, &c. therefore, ſuppoſing n to repreſent the number of quantities, and r the exponent of the combination, then ſhall $\overline{n-r+1}\cdot\frac{\overline{n-r+2}}{2}\cdot\frac{\overline{n-r+3}}{3}\cdot\frac{\overline{n-r+4}}{4}$ &c. to $\frac{n}{r}$ repreſent the combinations of r quantities in n: That is,

$\overline{n-r+1}\cdot\frac{n}{r}$ combinations of two quantities in n.

$\overline{n-r+1}\cdot\frac{\overline{n-r+2}}{2}\cdot\frac{n}{r}$ ———— three quantities in n.

$\overline{n-r+1}\cdot\frac{\overline{n-r+2}}{2}\cdot\frac{\overline{n-r+3}}{3}\cdot\frac{n}{r}$ ——four quantities in n.

&c. &c.

116. The ſums of all the combinations found by taking 0, 1, 2, 3, 4, *&c.* quantities out of any number (n) of things propoſed, is called the *election* of quantities, and is expreſſed univerſally by $\dot{2}$, or excluding 0 by $\dot{2}-1$, or excluding both 0, 1 by $\dot{2}-\overline{n+1}$.

117. Corol. 9. By taking the reciprocals of the terms of an arithmetical progreſſion, we obtain a ſeries of quantities, in continued HARMONICAL, OR MUSICAL PROPORTION; and by taking the difference of every two contiguous terms of the harmonical ſeries, the LAW of this ſeries will be evident by inſpection, Thus:

Arithmetical ſeries. $a,\ a\pm d,\ a\pm 2d,\ a\pm 3d,\ a\pm 4d,\ a\pm 5d, \ldots \ldots a\pm\overline{n-2}\times d,\ a\pm\overline{n-1}\times d$

Harmonical ſeries. $\frac{1}{a},\ \frac{1}{a\pm d},\ \frac{1}{a\pm 2d},\ \frac{1}{a\pm 3d},\ \frac{1}{a\pm 4d},\ \frac{1}{a\pm 5d}, \ldots\ldots \frac{1}{a\pm\overline{n-2}\times d},\ \frac{1}{a\pm\overline{n-1}\times d},$

Difference of the terms of Harmonical ſeries.

$$\left.\begin{array}{l} \frac{1}{a}\cdot\frac{d}{a\pm d},\ \frac{1}{a\pm d}\cdot\frac{d}{a\pm 2d},\ \frac{1}{a\pm 2d}\cdot\frac{d}{a\pm 3d}, \\ \frac{1}{a\pm 3d}\cdot\frac{d}{a\pm 4d},\ \frac{1}{a\pm 4d}\cdot\frac{1}{a\pm 5d}, \ldots\ldots \\ \frac{1}{a\pm\overline{n-2}\times d},\cdot\frac{d}{a\pm\overline{n-1}\times d}. \end{array}\right.$$

From the differential ſeries it appears directly, that in the harmonical progreſſion, there is always the ſame *ratio* between the two differences of any three contiguous terms, as there is between the two extremes of the ſaid three contiguous terms; alſo, that there is the ſame *ratio* between the product of the two firſt terms of the harmonical ſeries, and the product of any two other contiguous terms, as there is between the two differences of ſaid contiguous terms. And hence,

Suppoſing a ſeries of harmonical quantities repreſented by } A, B, C, D, E, F, ... V

And the ſeries of differences by M, N, P, Q, R, ... X

Then A=B±M, B=C±N, C=D±P, D=E±Q, E=F±R, *&c.*

Therefore, the ratio $M : N = B \pm 1 M : C$

—— $M : P = B \pm 2 M : D$

—— $M : Q = B \pm 3 M : E$

—— $M : R = B \pm 4 M : F$

&c. *&c.*

And univerſally, the ratio $M : X = B \pm \overline{n - 2} . M : V$

Thus, in any ſeries of quantities in harmonical or muſical progreſſion, there is the ſame *ratio* between the difference of the two firſt terms, and the difference of any two other contiguous terms, as there is between the difference or ſum of the ſecond term, and ſuch a multiple of the difference of the two firſt terms, as is denoted by the number of intermediate terms, and the laſt term of the ſeries.

118. All the terms of a continued harmonical ſeries may be derived from the two firſt terms only. Thus A being the firſt term, B the ſecond, and M their difference; ſince $A = \frac{A \times B}{B}$, and $B \left(= \frac{A \times B}{A}\right) = \frac{A \times B}{B \pm M}$ (117) therefore the harmonical progreſſion will be repreſented by the following ſeries of terms, viz.

$$\frac{A \times B}{B} (= A), \frac{A \times B}{B \pm M}, \frac{A \times B}{B \pm 2 M}, \frac{A \times B}{B \pm 3 M}, \frac{A \times B}{B \pm 4 M}, \ldots . \frac{A \times B}{B \pm \overline{n - 2} . M}$$

And from this ſeries it is obvious, that a harmonical progreſſion, conſiſting of *any whatever* number of terms, cannot always be derived from *any* two numbers or quantities A, B taken at pleaſure.

119. As three quantities, A, B, C, are in continued harmonical or muſical progreſſion, when the ratio $A \backsim B : B \backsim C = A : C$ (117): ſo, four quantities, A, B, C, D, are *ſaid* to be in harmonical or muſical proportion; when the ratio $A \backsim B : C \backsim D = A : D$, and three quantities, A, B, C, are ſaid to be in *contra-harmonic* proportion, when the ratio $A \backsim B : B \backsim C = C : A$. Thus,

Theſe three numbers, 10, 15, 30, are in continued harmonical proportion. Theſe four numbers, 10, 16, 24, 60, are in harmonical proportion; and theſe three 6, 5, 3, or theſe 12, 10, 4, are in *contra-harmonic* proportion.

120. Corol. 10. From the arithmetical progreſſion of odd numbers, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, *&c.* which may be continued for ever, we have ſhewn how all powers of the numbers in the natural progreſſion, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *&c.* may be produced by addition only (107). Further, this ſame progreſſion of odd numbers, indefinitely extended, contains all Prime Numbers, except the number 2; and the law of the progreſſion is ſuch, that *between every two multiples of any number N, as they ſtand in their natural order in the ſeries, there conſtantly intervene N—1 numbers, which are not multiples of N.* Hence all prime numbers, except the number 2, may be obtained from the progreſſion of odd numbers, by the following operation, which Dr Horsly † takes to be that called

The Sieve of Eratosthenes.

Let the ſeries of odd numbers be continued indefinitely,

1, 3, 5, 7, $\dot{9}$, 11, 13, $\dot{15}$, 17, 19, $\dot{21}$, 23, $\dot{25}$, $\dot{27}$, 29, 31, $\dot{33}$, $\dot{35}$, 37, $\dot{39}$, 41, 43, $\dot{45}$, 47, $\dot{49}$, $\dot{51}$, 53, $\dot{55}$, $\dot{57}$, 59, 61, $\dot{63}$, $\dot{65}$, 67, $\dot{69}$, 71, 73, $\dot{75}$, $\dot{77}$, 79, $\dot{81}$, 83, $\dot{85}$, $\dot{87}$, 89, $\dot{91}$, $\dot{93}$, $\dot{95}$, 97, $\dot{99}$, 101, 103, $\dot{105}$, 107, 109, $\dot{111}$, 113, $\dot{115}$, $\dot{117}$, $\dot{119}$, $\dot{121}$, $\dot{123}$, $\dot{125}$, 127, $\dot{129}$, 131, $\dot{133}$, $\dot{135}$, 137, 139, $\dot{141}$, $\dot{143}$, $\dot{145}$, $\dot{147}$, 149, 151, $\dot{153}$, $\dot{155}$, 157, $\dot{159}$, $\dot{161}$, 163, $\dot{165}$, *&c. &c.*

Then, (1.) Count all the terms of the ſeries following the number 3 by threes, expunging every third number; and thus, all the multiples of 3 will be exterminated. ‡ (2.) The firſt uncancelled number in the ſeries after 3 being 5, expunge its ſquare (25), and count all the terms of the ſeries, which follow the ſquare of 5 by fives, expunging every fifth number, if not expunged before; and thus, all the multiples of 5 will be exterminated, which were not at firſt exterminated among the multiples of 3. (3.) The next uncancelled number in the ſeries after 5 being 7, expunge its ſquare (49), and count all the terms of the ſeries which follow the ſquare of 7, by ſevens, expunging every ſeventh number if not expunged before; and thus, all the multiples of 7 will be exterminated, which were not before exterminated among the multiples of 3 and 5. (4.) The next uncancelled number in the ſeries after 7 being 11, expunge its ſquare (121), and count all the terms of the ſeries which follow the ſquare of 11 by elevens, expunging every eleventh number, if not expunged before; and thus, all the multiples of 11 will be exterminated, which were not before exterminated among the multiples of 3, 5 and 7. (5.) Laſtly, continue theſe expunctions 'till the firſt uncancelled number that appears, next to that whoſe multiples have been laſt expunged, is ſuch, that its ſquare is greater than the laſt and greateſt number, to which the ſeries is extended. The numbers which then remain uncancelled, are all the prime numbers, except the number 2, which occur in the natural progreſſion of numbers, from 1 to the limit of the ſeries.

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† Philoſophical Tranſactions, vol. 62. p. 327.

‡ The numbers expunged have a point placed over them.

Thus, the following numbers are all primes, *viz.*

1, [2], 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, *&c.*

121. The natural progreſſion 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *&c.* indefinitely extended, comprehending *all numbers*, prime, compoſite, *&c.* let n repreſent each term of the ſeries of natural numbers, indefinitely extended; then, $n \times 6 - 1$ and $n \times 6 + 1$ will conſtitute a ſeries comprehending all prime numbers, except the numbers 2 and 3.

122. Corol. 11. General theorems, reſolving all the caſes of Simple Interest, relative to annuities, penſions, rents, *&c.* in arrear or reverſion, may be derived by ſumming the terms of an arithmetical progreſſion Thus, ſuppoſing p to repreſent any principal ſum lent out for the time t, and that r is the *ratio* of the rate, or the intereſt of 1 *l.* for one year, then the intereſt of any principal p, being manifeſtly in the compound ratio of the principal p, time t, and rate r; therefore, ſuppoſing that s is the amount of the principal p, and its intereſt $t\,r\,p$ for the time t, all the varieties in this caſe are reſolved from the general equation $\overline{1+rt} \times p = s$, as in the following table.

p, r, t	$s = \overline{1+rt} \times p$
s, r, t	$p = \frac{s}{1+rt}$
s, r, p	$t = \frac{s-p}{rp}$
s, p, t	$r = \frac{s-p}{tp}$

123. Suppoſe now a to repreſent any annuity, penſion, rent, *&c.* in *arrear;* then, ſince 1 *l.* hath the ſame abſolute *ratio* to its rate (r) as any annuity, *&c.* (a) to its rate ($a r$); therefore,

$a \times 1$ 1ſt year's amount.
$a \times \overline{1 + r}$ 2d year's amount.
$a \times \overline{1 + 2r}$ 3d year's amount.
$a \times \overline{1 + 3r}$ 4th year's amount.
&c. *&c.*

Univerſally, $a \times \overline{1 + \overline{t-1} \times r}$ amount for time, t.

But the number of terms of this ſeries being t, the ſum s is $t a \times \overline{1 + \frac{t-1}{2} r}$ (101, 103); and from this equation, all the varieties in this caſe are reſolved as in the following table.

a, t, r	$s = t a \times \overline{1 + \frac{t-1}{2} r}$
s, t, r	$a = \frac{s}{t} \times \frac{1}{1 + \frac{t-1}{2} r}$
s, t, a	$r = \frac{\overline{\frac{s}{t} - a} \times 2}{\overline{t-1} \times a}$
s, r, a	$t = \sqrt{\frac{2s}{ar} + \overline{\frac{1}{r} - \frac{1}{2}}^{2}} - \overline{\frac{1}{r} - \frac{1}{2}}$

124. But if a repreſents any annuity, penſion, rent. *&c.* in *reverſion,* then, ſince the amount of 1 *l.* (122) for any time, ($1 + t r$) hath the ſame abſolute *ratio* to 1 *l.* as the amount (123) of any annuity, *&c.* ($t a \times \overline{1 + \frac{t-1}{2} r}$) to the preſent value ($s$) of ſaid annuity, *&c.* therefore all the varieties in this caſe, will be reſolved as in the following table.

a, r, t	$s = ta \times \dfrac{1 + \frac{t-1}{2} r}{1 + tr}$
s, r, t	$a = \dfrac{s}{t} \times \dfrac{1 + tr}{1 + \frac{t-1}{2} r}$
s, t, a	$r = \dfrac{\frac{s}{t} - a}{s + \frac{t-2}{2} a}$
s, r, a	$t = \sqrt{\frac{2s}{ra} + \overline{\frac{s}{a} + 1}^{\,2}} + \overline{\frac{s}{a} + 1}$

115. In computing the intereſt of money for any number of days, not exceeding a year, at 5 *l. per cent.* the following demonſtrative practical method may be uſeful: Let the product of the principal ſum by the number of days be advanced four places to the right hand; multiply this number by 3, advancing the product one place to the right hand; ſubtract the latter advanced product from the former, advancing the remainder one place to the right hand; then, the ſum of the two advanced products, and advanced remainder, diminiſhed by itſelf, when advanced four places to the right hand, will be the true intereſt required.

Thus, to find the intereſt of 4239 *l.* 15 *s.* 4 *d.* (=4239·767) for 256 days.

	4239·767×256=1085380.352 advanced 4 places	108.53803
	108.53803×3= 325. 61409 advanced 1 place	32.56151
From	108.53803 } 75·97662	
Take	32.56141 } Remainder advanced 1 place	7·59766
	From the ſum . . .	148.6971
	Take itſelf advanced 4 places . .	0.0148
	Remains the intereſt required . .	148.6822

That is, 148 *l.* 13 *s.* 7 ¾ *d.*

126. Intereſt-table for any ſum of money, from 1 million to 1 *l.* for any number of days, and at any rate.

Number.	*L.*	*S.*	*D.*	*Q.*	Numb.	*L.*	*S.*	*D.*	*Q.*
1000000	2739	14	6	0,99	1000	2	14	9	2,14
900000	2465	15	0	3,29	900	2	9	3	3,12
800000	2191	15	7	1,59	800	2	3	10	0,11
700000	1917	16	1	3,89	700	1	18	4	1,10
600000	1643	16	8	2,19	600	1	12	10	2,80
500000	1369	17	3	0,49	500	1	7	5	3,70
400000	1095	17	9	2,79	400	1	1	11	0,50
300000	821	18	4	1,09	300	0	16	5	1,40
200000	547	18	10	3,40	200	0	10	11	2,30
100000	273	19	5	1,70	100	0	5	5	3,01
90000	246	11	6	0,32	90	0	4	11	0,71
80000	219	3	6	0,96	80	0	4	4	2,41
70000	191	15	7	1,59	70	0	3	10	0,11
60000	164	7	8	0,22	60	0	3	3	1,81
50000	136	19	8	2,85	50	0	2	8	3,51
40000	109	11	9	1,48	40	0	2	2	1,21
30000	84	3	10	0,11	30	0	1	7	0,90
20000	54	15	10	2,74	20	0	1	1	0,60
10000	27	7	11	1,37	10	0	0	6	2,30
9000	24	13	1	3,23	9	0	0	5	3,67
8000	21	18	4	1,10	8	0	0	5	1,04
7000	19	3	6	2,96	7	0	0	4	2,41
6000	16	8	9	0,82	6	0	0	3	3,78
5000	13	13	11	2,58	5	0	0	3	1,15
4000	10	19	2	0,55	4	0	0	2	2,52
3000	8	4	4	2,41	3	0	0	1	3,89
2000	5	9	7	0,27	2	0	0	1	1,26
1000	2	14	9	2,14	1	0	0	0	2,63

To find the intereſt of any ſum of money by the table, multiply the principal ſum by the number of days, and this product by the rate *per cent.*; cut off the two laſt figures to the right hand, and find the other numbers in the table.

Thus, in the former example (123) $4239{\cdot}767\times 5^{\times 256}_{\wedge} = 54269|0176$

Then 50000	L. 136	19	8	~~[illegible]~~ 2.85
4000	10	19	2	~~[illegible]~~ 0.55
200	0	10	11	~~[illegible]~~ 2.30
60	0	3	3	~~[illegible]~~ 1.81
9	0	0	5	~~[illegible]~~ 3.64
Intereſt required.	L. 148	13	7	~~[illegible]~~ 3.18

Proposition XXIX.

127. If we aſſume any quantity at pleaſure, and therefrom derive a ſeries of terms, by *equal multipliers or diviſors*; the terms of the ſeries are ſaid to be in **Continued Geometrical Progression** (99): And the ſum of all the terms of the ſeries, except the laſt term, will have the ſame *ratio* to the ſum of all the terms, except the firſt term; as the common multiplier or its reciprocal to Unity.

Demonstration.

Let a, v, repreſent the two extreme terms in a continued geometrical progreſſion, whereof r or $\frac{1}{r}$ is the common multiplier, n the number, and s the ſum of all the terms: Then, according as we take a or v for the leading quantity, the ſeries will be expreſſed by the following terms, *viz.*

$$\text{A.}\quad a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots\ldots\ ar^{n-1}\quad (=v)$$

$$\text{B}\left\{\begin{array}{l} v,\ \dfrac{v}{r},\ \dfrac{v}{r^2},\ \dfrac{v}{r^3},\ \dfrac{v}{r^4},\ \ldots\ldots\ \dfrac{v}{r^{n-1}}\quad (=a) \\ \text{or (8)}\ \overset{v,}{{}_\wedge}\ vr^{-1},\ vr^{-2},\ vr^{-3},\ vr^{-4},\ \ldots\ldots\ vr^{-\overline{n-1}}\quad (=a) \end{array}\right.$$

And from theſe ſeries, it appears by inſpection, that $s-a = \overline{s-v}\times r$ (A) or $s-a = \overline{s-v}\times\frac{1}{r}$ (B); and therefore (16) the *ratio* $s-a : s-v = r : 1$ (A), or the *ratio* $s-a : s-v = \frac{1}{r} : 1$ (B).

Q. E. D.

128. Corol. 1. Since $ar^{n-1} = v$, or $vr^{-\overline{n-1}} = a$ (127) therefore, the *ratio* $1 : r^{n-1} = a : v$, or the *ratio* $1 : r^{-\overline{n-1}} = v : a$ (16). Thus: in any continued geometrical progreſſion, there is the ſame *ratio* between unity and that power of the com-

mon multiplier or diviſor, whereof the exponent is the exceſs of the number of terms above unity, as there is between the two extreme terms.

129. Corol. 2. Hence, in a continued geometrical progreſſion, if *any three* of the five quantities a, v, r, n, and s, are ſuppoſed to be GIVEN, the remaining *two* may be determined, as in the following table, where the expreſſions belong to an increaſing progreſſion, and may be applied alſo to a decreaſing progreſſion, by making the quantities a and v change places every where.

Caſes	*Data*	*Quæſita*	DETERMINATION. [Suppoſe $r^{n-1} + r^{n-2} + r^{n-3} + \ldots + 1 = Q$]
I.	a, r, n	v	$= a\, r^{n-1}$
		s	$= a \times \frac{r^n - 1}{r - 1} = a \times Q.$
II.	a, r, v	n	$= \frac{L,\, v - L,\, a}{L,\, r} + 1.$
		s	$= \frac{r\, v - a}{r - 1} = \frac{v - a}{r - 1} + v.$
III.	a, r, s	n	$= \frac{L,\, \overline{\overline{r-1} \times s + a} - L,\, a}{L,\, r}.$
		v	$= \frac{a + \overline{r-1} \times s}{r} = s - \frac{s-a}{r}.$
IV.	a, v, s	n	$= \frac{L,\, v - L,\, a}{L,\, \overline{s-a} - L,\, \overline{s-v}} + 1.$
		r	$= \frac{s-a}{s-v} = \overline{s-a} \times \frac{1}{s-v}.$
V.	a, n, s	v	Equation $s \times \overline{v}\vert^{\frac{1}{n-1}} - \overline{v}\vert^{\frac{n}{n-1}} = \overline{a}\vert^{\frac{1}{n-1}} \times \overline{s-a}.$
		r	Equation $\frac{s}{a} r - r^n = \frac{s}{a} - 1.$

VI.

VI.	a, n, v	r	$= \overline{\frac{v}{a}}\Big\vert^{\frac{1}{n-1}}$
		s	$= \frac{v-a}{\overline{\frac{v}{a}}\Big\vert^{\frac{1}{n-1}} - 1} + v$
VII.	r, v, n	a	$= \frac{v}{r^{n-1}} = v r^{-\overline{n-1}}.$
		s	$= \frac{v}{r^{n-1}} \times \frac{r^n-1}{r^n-1} = \frac{v}{r^{n-1}} \times Q = Q \times v r^{-\overline{n-1}}.$
VIII.	r, v, s	a	$= rv - s \times \overline{r-1}$
		n	$= \frac{L, v - L, \overline{rv - s \times \overline{r-1}}}{L, r} + 1.$
IX.	r, n, s	a	$= s \times \frac{r-1}{r^n-1} = s \times \frac{1}{Q} = \frac{s}{Q}.$
		s	$= r^n-1 \; s \times \frac{r-1}{r^n-1} = r^{n-1} \times \frac{s}{Q}.$
X.	n, v, s	a	Equation $s \times \overline{a}\vert^{\frac{1}{n-1}} - \overline{a}\vert^{\frac{n}{n-1}} = \overline{v}\vert^{\frac{1}{n-1}} \times \overline{s-v}.$
		r	Equation $\frac{s}{s-v} r^{n-1} - r^n = \frac{v}{s-v}.$

130. Corol. 3. As the quantities *a*, *v* may change places every-where (127, 129;) therefore, in any continued geometrical progreſſion, ſince the ratio $1 : r = s - v : s - a$ if we always expound *r* (which is generally called *the common ratio of the ſeries*) by the quotient of the greater by the leſſer contiguous term, we ſhall have univerſally the ratio $1 : r - 1 = s - v : v \backsim a$ (13). Thus: in any continued geometrical progreſſion, there is the ſame *ratio* between unity, and the exceſs of the common ratio above unity, as there is between the exceſs of the ſum of all the terms of the ſeries above the greateſt or leaſt term, and the difference of the two extreme terms.

131. Corol. 4. From the geneſis of a continued geometrical progreſſion, it appears, that one of the extreme terms, *repeated* as often as there are units in the other extreme term, is always

equal to any immediate term, at any diſtance from one of the extremes, *repeated* as often as there are units in the intermediate term, equi-diſtant from the other extreme; and if the number of terms be odd, then one extreme term, repeated as often as there are units in the other extreme term, is always equal to the ſquare or ſecond power of the middle term.

132. Corol. 5. Between the adjacent terms of any continued geometrical progreſſion, there always obtain *equal ratios;* and the ratio of the two extreme terms is ſuch a multiple of the common ratio of the ſeries, as is denoted by the number of equal ratios, from one extreme term to the other.

Thus, in the ſeries a, ar, ar^2, ar^3, ar^4, ar^5, . . . ar^{n-2}, ar^{n-1}, the ratios $a : ar$, $ar : ar^2$, $ar^2 : ar^3$, $ar^3 : ar^4$, $ar^4 : ar^5$ . . . $ar^{n-2} : ar^{n-1}$ are all equal; as being each equivalent to the ratio $1 : r$, which is therefore the common ratio of the ſeries; and the ratio $a : ar^{n-1} = 1 : r^{n-1} = \overline{n-1} \times \overline{1 : r}$ (5).

133. Corol. 6. By ſtating the equal ratios between the adjacent terms of a continued geometrical progreſſion, as in the laſt corollary (132); it is obvious, that $s-a$ is the ſum of all the antecedent, and $s-v$ that of all the conſequent terms of the ſeries (127); and therefore, in any continued geometrical progreſſion, the ratio between any two adjacent terms, taken in ſucceſſion, is always equal to the ratio between the ſum of all the antecedent, and that of all the conſequent terms of the ſeries (15).

134. Corol. 7. If out of any ſeries of quantities in continued geometrical progreſſion, there be taken any ſeries of equi-diſtant terms, that ſeries will alſo be in continued geometrical progreſſion. Thus, let the propoſed geometrical progreſſion be,

$$a,\ ar,\ ar^2,\ ar_3,\ ar^4,\ ar^5,\ ar^6,\ ar^7,\ ar^8,\ ar^9,\ ar^{10},\ \&c.$$

Then, the ſeries $\left\{\begin{array}{l} a,\ ar^2,\ ar^4,\ ar^6,\ ar^8,\ ar^{10},\ \&c. \\ a,\ ar^3,\ ar^6,\ ar^9,\ \&c. \\ ar,\ ar^3,\ ar^5,\ ar^7,\ ar^9,\ \&c. \\ \quad \&c. \qquad\qquad\qquad \&c. \end{array}\right.$

are all in continued geometrical progreſſion (127).

135. Corol 8. Suppoſing the leading quantity a of the general ſeries a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, &c. equal to unity; it is obvious, that the terms 1, r, r^2, r^3, r^4, r^5, r^6, &c. are in the ſame continued geometrical progreſſion as before; and therefore, by the laſt corollary (134) the following ſeries are all in continued geometrical progreſſion, *viz.*

$$1,\ r^2,\ r^4,\ r^6,\ r^8,\ r^{10},\ \&c.$$
$$1,\ r^3,\ r^6,\ r^9,\ r^{12},\ r^{15},\ \&c.$$
$$1,\ r^4,\ r^8,\ r^{12},\ r^{16},\ r^{20},\ \&c.$$

Univerſally, $1,\ r^m,\ r^{2m},\ r^{3m},\ r^{4m},\ r^{5m},\ \&c.$

Thus: in any continued geometrical progreſſion, if the firſt term be any power, then all the other terms of the ſeries are ſimilar powers, or powers under the ſame exponent.

136. Corol. 9. The ſeries a, ar, ar^2, ar^3, dr^4, ... ar^{n-1}, and a, ar^{-1}, ar^{-2}, ar^{-3}, ar^{-4}, . . dr^{-n-1}, may be conſidered as derived, by repeating the terms of a ſeries of equal quantities, a, a, a, a, a . . . . . a as often as there are units in the correſponding terms of the continued geometrical progreſſions 1, r, r^2, r^3, r^4, ... r^{n-1} and 1, r^{-1}, r^{-2}, r_{-3}, r^{-4}, ... r^{-n+1} : and hence it appears, that the common ratio of any continued geometrical progreſſion is not changed, by taking any equimultiples, integral or fractional, of all the terms of the progreſſion (132.) The moſt ſimple continued geometrical progreſſions, therefore, are ſuch, whereof the firſt or initial term is unity. Thus, ſuppoſing m to repreſent any number, the following are the moſt ſimple continued geometrical progreſſions, in the ratio of 1 to r^m or r^{-m}.

$$1,\ r^{m},\ r^{2m},\ r^{3m},\ r^{4m},\ r^{5m},\ r^{6m},\ \&c.$$
$$1,\ r^{-m},\ r^{-2m},\ r^{-3m},\ r_{-4}^{m},\ r^{-5m},\ r^{-6m},\ \&c.$$

or (7, 8)

$$r^{0},\ r^{m},\ r^{2m},\ r^{3m},\ r^{4m},\ r^{5m},\ r^{6m},\ \&c.$$
$$r^{0},\ r^{-m},\ r^{-2m},\ r^{-3m},\ r^{-4m},\ r^{-5m},\ r^{-6m},\ \&c.$$

In theſe ſeries, the exponents of the terms are evidently in continued arithmetical progreſſion (101); and the *ratios* of the terms of the geometrical progreſſion, to the firſt term, are denoted by theſe exponents; or, the exponents are LOGARITHMS of the correſponding terms of the geometrical progreſſion.

137. Corol. 10. Since, in a continued decreaſing geometrical progreſſion, the ratio $1 : r-1 = s-a : a-v$ (130); therefore, ſuppoſing v to vaniſh, or the decreaſing ſeries to have *no laſt term*, and in this caſe, the ratio $1 : r-1 = s-a : a$ which gives $s = \frac{ra}{r-1} =$ [illegible] $ra \times \frac{1}{1-r}$ (16): and this expreſſion $ra \times \frac{1}{r-1}$ is therefore the LIMIT of the ſum, ſuppoſing the ſeries continued *indefinitely*; that is, ſuppoſing the ſeries continued to a number of terms, *greater* than any aſſignable number. For, if we reſolve the expreſſion $ra \times \frac{1}{r-1}$ by diviſion (7); or by Sir Iſaac Newton's binomial theorem (78), we ſhall obtain directly the propoſed decreaſing ſeries, *viz.* $a + \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} + \frac{a}{r^4}$, &c. or

$a \times \overline{1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4}}$ &c. which ſeries may be continued for ever; and as the ſum of this ſeries varies continually, according as the number (n) of terms vary, ſo the varying ſum conſtantly approaches nearer and nearer, to the determinate quantity $ra \times \frac{1}{n-1}$, and may, by increaſing the number (n) of terms, be brought nearer to this quantity than by any aſſignable difference; but can never, by any acceſſion of terms, be made actually to reach, much leſs to paſs beyond it; for the determinate quantity $ra \times \frac{1}{r-1}$ ariſes only from the ſuppoſition of the ſeries having no laſt term, or from that of its being continued *indefinitely*. Here all the varieties which can take place are exhibited as in the following table, *viz.*

a, r	$s = ra \times \frac{1}{r-1}$
a, s	$r = s \times \frac{1}{s-a}$
r, s	$a = s \times \overline{1 - \frac{1}{r}}$

138. Corol. 11. Let a, b be the two initial terms of a ſeries decreaſing *indefinitely*; then, ſince $\frac{a}{b} = r$ (130) the LIMIT o the ſum is therefore $\frac{a^2}{a-b} = a^2 \times \frac{1}{a-b}$: and hence the ſeries itſelf is $a + b + \frac{b^2}{a} + \frac{b^3}{a^2} + \frac{b^4}{a^3} +$, &c. *indefinitely*; or, $a \times \overline{1 + \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + \frac{b^4}{a^4}} +$, &c. *indefinitely*; with which ſeries, any decreaſing geometrical progreſſion may be compared, and the LIMIT of the ſum of conſequence determined.

139. Corol. 12. Suppoſing n, m to repreſent numbers, then, from the terms of the continued geometrical progreſſion, n^m, n^{m-1}, n^{m-2}, n^{m-3}, n^{m-4}, ... n the COMPOSITION of quantities may be derived. Thus, the compoſition of m quantities in n is n^m, that of $m-1$ quantities in n is n^{m-1}, that of $m-2$

quantities in n is n^{m-2}, and ſo on; and hence, from the ſum of this continued geometrical progreſſion, (130) we ſhall have this expreſſion $n \times \frac{n^m - 1}{n - 1}$ or (taking $m = n$) $n \times \frac{n^n - 1}{n - 1}$ which denotes all the ſeveral compoſitions or changes there may be of n quantities, combined all poſſible ways, or taking them by one's, two's, three's, *&c.* to m or n. Thus, all the compoſitions or changes of 4 quantities is $(n^n =)\ 4^4 = 256$, taking each time 4; and therefore, $n \times \frac{n^n - 1}{n - 1} = 340$ is the number of all the compoſitions or changes of 4 quantities, combined all poſſible ways: Thus alſo, all the compoſitions or changes of 24 quantities, is $(n^n =)$ $24^{24} = 1333735776850284124449$, by taking each time 24; and therefore, the number of all poſſible compoſitions or changes, in this caſe, is $n \times \frac{n^n - 1}{n - 1} = 139172428887252999425128493402200$, and ſo many ways may the 24 letters of the alphabet be put together among themſelves.

140. Corol. 13. According to **Euclid** *, a number which is equal to all its *aliquot* parts taken together, is a **Perfect Number**. And *if, from unity, there be taken a ſeries of numbers in the ratio of* 1 *to* 2, *(*136*) continued to ſuch a number of terms, that the ſum of all the terms is a prime number, the product of this ſum, by the laſt term of the ſeries, will be a* perfect number †. Suppoſe, therefore, $s = 1 + 2 + 4 + 8 \ldots\ldots + 2^{n-1}$, where s is a prime number; then $2s = 2 + 4 + 8 + 16 \ldots\ldots + 2^{n-1} + 2^n$; and therefore, $(\underline{2s - s} =)\ s = 2^n - 1$: whence, according to **Euclid**, $2^{n-1}, \times 2^n - 1$, or $s \times 2^{n-1}$ is a perfect number, $2^n - 1$, or s, being a prime number. Since then $s = 2^n - 1$, therefore $s + 1 = 2^n = 2 \times 2^{n-1}$, (6), and $\frac{s+1}{2^{n-1}} = \frac{2}{1}$ (Ax. x); whence, by **Prop. XIX.** (40—44).

$$\begin{array}{ll} s + 1 = 2,\ 4,\ 6,\ 8,\ \&c. & \text{(com. Diff. 2.)} \\ 2^{n-1} = 1,\ 2,\ 3,\ 4,\ \&c. & \text{(Ditto 1.)} \end{array}$$

$$\text{Taking then, } \begin{array}{l} s + 1 = 2,\ 4 \\ 2^{n-1} = 1,\ 2 \end{array} \left\{ \begin{array}{l} s = 1,\ 3 \\ 2^{n-1} = 1,\ 2 \end{array} \right.$$

And from theſe two initial values of s and 2^{n-1} (which give the firſt two perfect numbers 1 and 6) all the other values may be obtained, by continuing the ſeries therefrom, (42); conſidering only that s muſt be a prime number, $n-1$ an even number, and of conſequence n an odd number. Thus,

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* Definition XXII. Book VII. † Euclid, Book IX. Prop. XXXVI.

$s = 1$	$1+2\times\overline{2^{1}-1}$	$1+2\times\overline{2^{2}-1}$	$1+2\times\overline{2^{4}-1}$	$1+2\times\overline{2^{6}-1}$	$1+2\times\overline{2^{8}-1}$	$1+2\times\overline{2^{10}-1}$	$1+2\times\overline{2^{12}-1}$	$1+2\times\overline{2^{14}-1}$	$1+2\times\overline{2^{16}-1}$	&c.
$2^{n-1}=1$	2	2^{2-1}	2^{4-1}	2^{6-1}	2^{8-1}	2^{6-1}	2^{12-1}	2^{14-1}	2^{16-1}	&c.
$n = 1$	2	3	5	7	9	11	13	15	17	&c.

This ſeries, which may be continued for ever, will aſſign all perfect numbers; and that in a more general and determinate manner, than any other ſeries which hath hitherto been given.

141. Corol. 14. Two numbers, A, B, are ſaid to be AMIABLE, when all the aliquot parts of A taken together make up the number B, and thoſe of B taken together make up the number A; ſo that amiable numbers may be conſidered as *interchangeable* perfect numbers. And univerſally, if 2^n, (where n is an affirmative integer), be taken ſuch, that $3\times 2^n-1$, $6\times 2^n-1$ and $18\times 2^{2n}-1$ be prime numbers; then, $2^{n+1}\times 18\,.\,2^{2n}-1$ ſhall be an amiable number, and $2^{n+1}\times 3\,.\,2^n-1\times 6\,.\,2^n-1$ its partner; or, ſuppoſing $2^n=Q$, and that $3Q-1$, $6Q-1$ and $18Q^2-1$ are prime numbers, then $2Q\times\overline{18Q^2-1}$ and $2Q\times\overline{3Q-1}\times\overline{6Q-1}$, are numbers amiable to one another. Here expounding n by 1, 3, 6, the firſt three pairs of amiable numbers are 284, and 220, 18416 and 17296, 9437050, and 9363584. And this general theorem, according to SCHOOTEN * was firſt given by the celebrated DES CARTES.

142. Corol. 15. General theorems, reſolving all the caſes of COMPOUND INTEREST, relative to annuities, penſions, rents, &c. in arrear, reverſion, or perpetuity, may be derived by ſumming the terms of a continued geometrical progreſſion. Suppoſe p to repreſent any principal ſum lent out for the time t, r the rate, or the intereſt of 1 *l.* for one year, and R $(=1+r)$ the amount of 1 *l.* and its intereſt for one year (called here the ratio of the rate); then, ſince the ratio $1 : R = R : R^2 = R^2 : R^3 = R^3 : R^4$, &c. (136) therefore R^t will univerſally be the amount of 1 *l.* for any time denoted by t: whence, ſince the amounts are in the ſame *ratio* with the principal ſums, therefore, ſuppoſing s to be the amount of the principal p, and its intereſt $p\,R^t$ for the time t,

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* *Sectiones Miſcellaneæ*, p. 423.

all the varieties in this caſe are reſolved from the general equation $p\,R^t = s$, as in the following table :

p, R, t	$p\,R^t = s$ or $L, p + t \times L, R = L, s$
s, R, t	$p = \frac{s}{R^t}$ or $L, p = L, s - t \times L, R$
s, R, p	$t = \frac{L, s - L, p}{L, R}$
s, p, t	$R = \sqrt[t]{\frac{s}{p}}$, or $L, R = \frac{L, s - L, p}{t}$

143. Suppoſe now *a* to repreſent any annuity, penſion, rent, *&c.* in *arrear ;* then, ſince the laſt year's annuity, *&c.* bears no intereſt, as tarying out no *time*, the firſt year's annuity, *&c.* will therefore univerſally be repreſented by $a\,R^{t-1}$: whence, deriving a ſeries by this propoſition (127), from the quantity *a* in the ratio $1 : R$ we ſhall have $a + a\,R + a\,R^2 + a\,R^3 \ldots\ldots + a\,R^{t-1} = s$, or $s = a \times \frac{R^t - 1}{R - 1}$ (129, 130 ;) and from this general equation, all the varieties in this caſe are reſolved as in the following table:

a, t, R	$s = a \times \frac{R^t - 1}{r}$ (142).
s, t, R	$a = r\,s \times \frac{1}{R^t - 1}$
s, t, a	$\frac{s}{a} R - R^t = \frac{s}{a} - 1$
s, R, a	$t = \frac{L, \overline{\frac{rs}{a} + 1}}{L, R}$

144. But, if *a* repreſents an annuity, penſion, rent, *&c.* in *reverſion*, then, ſince the ratio $1 : \frac{1}{R} = a : \frac{a}{R}$ the preſent worth of the annuity, *&c.* *a* at the end of one year will therefore be $\frac{a}{R}$: whence, deriving a ſeries by this propoſition (127) from the

quantity $\frac{a}{R}$ in the ratio $1 : \frac{1}{R}$ we ſhall have $\frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \frac{a}{R^4}$ $\ldots\ldots + \frac{a}{R^t} = s$ or $s = a \times \frac{1-\frac{1}{R^t}}{R-1}$ (129, 130); and from this equation, all the varieties in this caſe are reſolved as in the following table:

a, t, R	$s = a \times \frac{1-\frac{1}{R^t}}{r}$ (142)
s, t, R	$a = rs \times \frac{1}{1-\frac{1}{R^t}}$
s, t, a	$\overline{1+\frac{a}{s}} \times R^t - R^{t+1} = \frac{a}{s}$
s, R, a	$t = \frac{L, a - L, \overline{a - rs}}{L, R}$

145. If we ſuppoſe the progreſſion $\frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \frac{a}{R^4} \ldots$ $\ldots + \frac{a}{R^t}$ (144.) to have *no laſt term;* then, on this ſuppoſition, a repreſenting a PERPETUITY, or the yearly rent of a FREE-HOLD ESTATE, all the varieties in the buying and ſelling of eſtates in fee-ſimple, may be reſolved from the equation $s = a \times \frac{1}{R-1}$ (137) as in the following table:

a, R	$s = a \times \frac{1}{r}$
s, R	$a = rs$
s, a	$R = 1 + a \times \frac{1}{s}$

146. And if we ſuppoſe N to repreſent the number of years purchaſe a is worth, then, $N a$ $(= s)$ being the worth itſelf of a, therfore, inſtead of a ſubſtituting its value $(r s)$ from the fee-ſimple equation (145), and we ſhall have $N r = 1$. Thus, the

number (N) of years purchaſe a fee-ſimple, is worth at the rate R of compound intereſt is $\frac{1}{r}$; and the rate (R) of compound intereſt, at which the purchaſe-money of a fee-ſimple is valued for N years purchaſe is $1 + \frac{1}{N}$.

147. Laſtly, ſince $\frac{1}{r}\left(=\frac{1}{R-1}\right)=\frac{1}{R}+\frac{1}{R^2}+\frac{1}{R^3}+\frac{1}{R^4}+$, *&c.* therefore, the ſum of the values of 1 *l.* for the *ſucceſſive years* of the life of a perſon of a given age, would be expreſſed by this ſeries, were there no contingency in the caſe. And to allow for contingency, a table of the probabilities of life muſt be uſed; and hence may be computed the following tables:

TABLE I. *by Mr Thomas Simpſon, ſhewing the value of an annuity of* 1 *l. upon one* LIFE.*

Age.	Years purchaſe, at 5 *per cent.*	Ditto, at 4 *per cent.*	Ditto, at 3 *per cent.*	Age.	Years purchaſe, at 5 *per cent.*	Ditto, at 4 *per cent.*	Ditto, at 3 *per cent.*
6	14.1	16.2	18.8	41	10.2	11.4	13.0
7	14.2	16.3	18.9	42	10.1	11.2	12.8
8	14.3	16.4	19.0	43	10.0	11.1	12.6
9	14.3	16.4	19.0	44	9.9	11.0	12.5
10	14.3	16.4	19.0	45	9.8	10.8	12.3
11	14.3	16.4	19.0	46	9.7	10.7	12.1
12	14.2	16.3	18.9	47	9.5	10.5	11.9
13	14.1	16.2	18.7	48	9.4	10.4	11.8
14	14.0	16.0	18.5	49	9.3	10.2	11.6
15	13.9	15.8	18.3	50	9.2	10.1	11.4
16	13.7	15.6	18.1	51	9.0	9.9	11.2
17	13.5	15.4	17.9	52	8.9	9.8	11.0
18	13.4	15.2	17.6	53	8.8	9.6	10.7
19	13.2	15.0	17.4	54	8.6	9.4	10.5
20	13.0	14.8	17.2	55	8.5	9.3	10.3
21	12.9	14.7	17.0	56	8.4	9.1	10.1
22	12.7	14.5	16.8	57	8.2	8.9	9.9
23	12.6	14.3	16.5	58	8.1	8.7	9.6
24	12.4	14.1	16.3	59	8.0	8.6	9.4
25	12.3	14.0	16.1	60	7.9	8.4	9.2
26	12.1	13.8	15.9	61	7.7	8.2	8.9
27	12.0	13.6	15.6	62	7.6	8.1	8.7
28	11.8	13.4	15.4	63	7.4	7.9	8.5
29	11.7	13.2	15.2	64	7.3	7.7	8.3
30	11.6	13.1	15.0	65	7.1	7.5	8.0
31	11.4	12.9	14.8	66	6.9	7.3	7.8
32	11.3	12.7	14.6	67	6.7	7.1	7.6
33	11.2	12.6	14.4	68	6.6	6.9	7.4
34	11.0	12.4	14.2	69	6.4	7.7	7.1
35	10.9	12.3	14.1	70	6.2	6.5	6.9
36	10.8	12.1	13.9	71	6.0	6.3	6.7
37	10.6	11.9	13.7	72	5.8	6.1	6.5
38	10.5	11.8	13.5	73	5.6	5.9	6.2
39	10.4	11.6	13.3	74	5.4	5.6	5.9
40	10.3	11.5	13.2	75	5.2	5.4	5.6

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* Simpſon's Select Exercifes, p. 260.

TABLE II. *ſhewing the value of an annuity of* 1 *l. upon one* Life, *ſuppoſing the decrements of life to be equal.**

Age.	Years purchaſe, at 5 *per cent.*	Ditto, at 4 *per cent.*	Ditto, at 3 *per cent.*	Age.	Years purchaſe, at 5 *per cent.*	Ditto, at 4 *per cent.*	Ditto, at 3 *per cent.*
8	14.54	16.79	19.74	43	11.43	12.68	14.19
9	14.61	16.89	19.87	44	11.29	12.50	13.96
10	14.61	16.89	19.87	45	11.14	12.32	13.73
11	14.54	16.79	19.74	46	10.99	12.14	13.49
12	14.48	16.69	19.60	47	10.84	11.94	13.25
13	14.41	16.61	19.47	48	10.68	11.75	13.01
14	14.34	16.51	19.33	49	10.52	11.55	12.76
15	14.27	16.41	19.19	50	10.35	11.34	12.51
16	14.19	16.31	19.05	51	10.18	11.14	12.26
17	14.12	16.21	18.91	52	9.99	10.92	11.99
18	14.05	16.11	18.76	53	9.82	10.70	11.73
19	13.97	15.99	18.61	54	9.63	10.48	11.46
20	13.89	15.89	18.46	55	9.44	10.25	11.18
21	13.81	15.79	18.31	56	9.24	10.02	10.90
22	13.73	15.67	18.15	57	9.04	9.77	10.62
23	13.64	15.55	17.99	58	8.83	9.53	10.33
24	13.56	15.44	17.83	59	8.61	9.28	10.03
25	13.47	15.32	17.66	60	8.39	9.02	9.73
26	13.38	15.19	17.49	61	8.16	8.75	9.42
27	13.28	15.07	17.33	62	7.93	8.49	9.12
28	13.19	14.95	17.15	63	7.69	8.21	8.79
29	13.09	14.82	16.98	64	7.44	7.92	8.46
30	12.99	14.68	16.80	65	7.18	7.63	8.13
31	12.89	14.55	16.62	66	6.92	7.33	7.79
32	12.79	14.41	16.44	67	6.43	7.03	7.45
33	12.77	14.27	16.25	68	6.36	6.71	7.09
34	12.57	14.13	16.06	69	6.07	6.39	6.74
35	12.45	13.98	15.86	70	5.78	6.07	6.38
36	12.33	13.83	15.67	71	5.47	5.73	6.01
37	12.21	13.68	15.47	72	5.15	5.38	5.63
38	12.09	13.52	15.26	73	4.83	5.03	5.25
39	11.97	13.36	15.05	74	4.49	4.67	4.85
40	11.84	13.19	14.84	75	4.14	4.29	4.45
41	11.71	13.03	14.63	76	3.78	3.91	4.01
42	11.58	12.86	14.41	78	3.42	3.52	3.63

* Dodſon's Repoſitory, vol. 2. p. 169.

Note. By increaſing the numbers in table II, by $\frac{2}{8}$ of a year's purchaſe, to 54 years of age, by $\frac{3}{9}$ from 54 to 70, and by $\frac{4}{9}$ from 70 years and upwards, we ſhall obtain the preſent values *nearly* of an annuity (SECURED BY LAND) of 1 l. *per annum*, on a ſingle life; ſuppoſing the decrements of life to be equal *.

148. In the caſe of a ſimple annuity, (147, Tables I. and II.) the heirs of the annuitant are not entitled to any thing, if the annuitant deceaſes before the term of payment; but in the caſe of an annuity ſecured by land, the heirs of the annuitant are entitled to receive a ſum of money proportional to the time elapſed between the laſt payment of the annuity, and the time of the death of the annuitant: Thus, the value of a life of 40 years of age, at 4 l. *per cent*. compound intereſt, is 11.5, by Table I. and 13.19 years purchaſe, by Table II. and the value of an annuity ſecured by land for the ſame age and rate, is 13.412 years purchaſe nearly: Thus alſo, the value of a life of 66 years of age, at 4 l. *per cent*. compound intereſt, is 7.3 by Table I. and 7.33 by Table II. and the value of an annuity ſecured by land, for the ſame age and rate, is 7.66 years purchaſe *nearly*.

PROPOSITION XXX.

149. If we take out *any* two of the homologous terms of an analogy, theſe two quantities, when conſidered by themſelves, are ſaid to conſtitute a GENERAL PROPORTION; one of theſe quantities being always *as* the other: And any one of the terms of a general proportion may be multiplied or divided by any number taken at pleaſure.

Suppoſe the ratio $Q : q = R : r$, then to expreſs the general proportion between Q and R, or between q and r, we put the quantities down in this manner, *viz*. Q ·· R and q ·· r; and, by this notation, we mean nothing more than that the quantity Q is always *as* the homologous quantity R, and the quantity q *as* the homologous quantity r: Here then the quantities R, r may be either different values of the analogous quantities Q, q or conſidering Q, q as proper quantities, the quantities R, r may be improper quantities, ſuppoſing them referred to the proper quantities Q, q as their meaſures. Suppoſing (as above) the ratio $Q : q = R : r$, it is evident, that the quantity R will be greater, equal or leſs than the analogous quantity r, according as the quantity Q is greater, equal or leſs than the analogous quantity q: and therefore, conſidering the quantities Q, R as variable, they will either both together equally increaſe, or both together equally decreaſe. Thus, in every analogy, the homologous terms have a mutual dependence upon one another; and hence the propriety of the expreſſion introduced on this ſubjeƈt by the illuſtrious * Sir Iſaac Newton, that any term of an analogy is always AS its proper homologous term. Since then, by by ſuppoſition, the ratio $Q : q = R : r$, and ſince the ratio $R : r = \frac{m}{n} R : \frac{m}{n} r$ where m, n may repreſent any numbers whatever; therefore, according to the notation explained in this propoſition, when $Q \cdot\cdot R$ it is alſo $Q \cdot\cdot \frac{m}{n} R$ (Ax. IV.)

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* See Dodſon's Repoſitory, Vol. III.

150. Corol. 1. Suppoſe the ratio $Q : q = R : r = S : s$, &c. then according to the notation explained in this propoſition we have $Q \cdot\cdot R$, $R \cdot\cdot S$, &c. as alſo $Q \cdot\cdot S$: Therefore, if one quantity Q is as another quantity R, and the quantity R as another quantity S, &c. then will the firſt quantity Q be as the laſt quantity S.

151. Corol. 2. Suppoſe the ratio $Q : q = R : r = S : s$; then, ſince the ratio $Q : q = R \pm S : r \pm s$ (13) as alſo $(2 \times \overline{Q : q} = \overline{R : r} + \overline{S : s}$ or) the ratio $Q^2 : q^2 = RS : rs$ (5) therefore, when $Q \cdot\cdot R$ and $R \cdot\cdot S$, it is $Q \cdot\cdot R \pm S$, as alſo $Q^2 \cdot\cdot RS$ or $Q \cdot\cdot \sqrt{RS}$. Thus, if one quantity Q be as a ſecond quantity R, and the ſecond quantity R as a third quantity S; then will the firſt quantity Q be as the ſum or difference of the ſecond and third quantities R, S, and alſo as the mean proportional between them (16.)

152. Corol. 3. Suppoſe the ratio $Q : q = R : r$; then ſince $\frac{m}{n} \overline{Q : q} = \frac{m}{n} \overline{R : r}$ (IX. Ax.) or the ratio $Q^{\frac{m}{n}} : q^{\frac{m}{n}} = R^{\frac{m}{n}} : r^{\frac{m}{n}}$ (5, 7): Therefore, when $Q \cdot\cdot R$ it is alſo $Q^{\frac{m}{n}} \cdot\cdot R^{\frac{m}{n}}$.—That is, both the terms of any general proportion may be anyhow equally involved or evolved.

153. Corol. 4. Suppoſe the ratio $Q : q = R : r$, and let $S : s$ be any other ratio; then, ſince $\overline{Q : q} \pm \overline{S : s} = \overline{R : r} \pm \overline{S : s}$ (VII, VIII, Ax.) or the ratio $QS : qs = RS : rs$ (3), as alſo the ratio $\frac{Q}{S} : \frac{q}{s} = \frac{R}{S} : \frac{r}{s}$ (4); therefore when $Q \cdot\cdot R$ it is alſo $QS \cdot\cdot RS$ and $\frac{Q}{S} \cdot\cdot \frac{R}{S}$, ſuppoſing S to be any variable quantity whatever. Thus, both the terms of any general proportion may be multiplied or divided by any variable quantity taken at pleaſure.

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* Principia, B. I. Lem. 10. Scholium.

154. Corol. 5. Let Q, R, S, T, *&c.* be any number of variable quantities, and ſuppoſe that $Q \cdot\cdot RS$ and that $S \cdot\cdot T$, then will $Q \cdot\cdot RT$: For, ſince, by ſuppoſition $S \cdot\cdot T$, therefore $RS \cdot\cdot RT$ (153) and of conſequence $Q \cdot\cdot RT$ (150). Thus, in any general proportion, inſtead of any variable quantity may be ſubſtituted any other *homologous* variable quantity.

155. Corol. 6. Let Q, R be two variable *homologous* quantities. Then, ſince by ſuppoſition, $Q \cdot\cdot R$, therefore, dividing both terms by R, we have $\frac{Q}{R} \cdot\cdot 1$ (153); but unity is a given invariable quantity, and therefore $\frac{Q}{R}$ is alſo a given invariable quantity, or $\frac{Q}{R}$ is always the ſame, let the quantities Q, R be any whatever variable quantities. Hence, in any general proportion, inſtead of any *invariable* quantity, we may always ſubſtitute unity.

156. Corol. 7. Suppoſe the ratio $Q : q = R : r$, and the ratio $S : s = T : t$; then, ſince $\overline{Q : q} \pm \overline{S : s} = \overline{R : r} \pm \overline{T : t}$ (VII, VIII Ax.) or the ratio $QS : qs = RT : rt$ (3) as alſo the ratio $\frac{Q}{S} : \frac{q}{s} = \frac{R}{T} : \frac{r}{t}$ (4): Therefore, when $Q \cdot\cdot R$ and $S \cdot\cdot T$ it is alſo $QS \cdot\cdot RT$ and $\frac{Q}{S} \cdot\cdot \frac{R}{T}$. That is, the terms of any general proportion may be multiplied or divided by the terms of any other general proportion.

157. Corol. 8. Suppoſe the ratio $Q : q = \frac{1}{V} : \frac{1}{v}$; then conſidering the homologous quantities, Q, $\frac{1}{V}$ as variable, they will either both together equally increaſe, or both together equally decreaſe: But, ſince V, $\frac{1}{V}$ are the reciprocals of one another, if the former increaſes the latter will diminiſh, and *vice verſa*. Therefore, in conſidering two variable quantities, to determine their general proportion, if one of the quantities increaſes while the other equally diminiſhes, the former will always be *reciprocally* as the latter.

158. Corol. 9. Suppoſe the ratio $Q : q = \overline{R : r} + \overline{S : s}$; then if the ratio $S : s$ be a ratio of equality, the ratio $Q : q = R : r$, and if the ratio $R : r$ be a ratio of equality, the ratio $Q : q = S : s$. Therefore, if $Q \cdot\cdot R$ when S is invariable, and

$Q \cdot\cdot S$

$Q \cdot\cdot S$ when R is invariable; then, when both the quantities R, S are variable, it is $Q \cdot\cdot RS$.

159. Corol. 10. And in the ſame manner it may be demonſtrated, that if $Q \cdot\cdot R$ when V is invariable, and $Q \cdot\cdot \frac{1}{V}$ when R is invariable; then when both the quantities V, R are variable, it is $Q \cdot\cdot R \times \frac{1}{V}$.

160. Corol. 11. Let there be any cauſe A which produces the effect E in the time T; then, ſince we can judge of a cauſe by the effect which it can produce in a given portion of time, it is evident, that every effect produced in time depends not only upon its immediate efficient cauſe, but alſo on the time wherein the cauſe acts or exerts itſelf; and hence, by increaſing or diminiſhing the time, the effect will be equally increaſed or diminiſhed, allowing the cauſe to remain the ſame; and by increaſing or diminiſhing the cauſe, the effect will be equally increaſed or diminiſhed, ſuppoſing the time to remain the ſame. Thus, then $E \cdot\cdot T$ when A is invariable, and $E \cdot\cdot A$ when T is invariable; therefore univerſally, or when both A, T are variable, it is $E \cdot\cdot AT$. That is, the effect produced is always in the compound ratio of the efficient cauſe and the time.

161. Corol. 12. Since then univerſally $E \cdot\cdot AT$, therefore $E \times \frac{1}{A} \cdot\cdot T$ and $E \times \frac{1}{T} \cdot\cdot A$. That is, the time of producing an effect is as the ſaid effect directly, and its efficient cauſe reciprocally; and any efficient cauſe is directly as the effect which it can produce, and reciprocally as the time.

162. Corol. 13. If we ſuppoſe the effect (E) to be invariable, then $AT \cdot\cdot 1$ (155) and of conſequence $A \cdot\cdot \frac{1}{T}$ or $T \cdot\cdot \frac{1}{A}$.—Thus, with regard to a *given* effect, the efficient cauſe is reciprocally as the time, or the time reciprocally as the efficient cauſe.

163. Corol. 14. In the general theorem (160) $E \cdot\cdot AT$, ſuppoſe that $A \cdot\cdot T$; then in this caſe, $E \cdot\cdot A^2$ or $E \cdot\cdot T^2$. That is, when the efficient cauſe is as the time, the effect produced will be in the duplicate ratio of the efficient cauſe, or in the duplicate ratio of the time.

FINIS.

NOTES.

A.—PROPOSITION I.

164. **Addition** being the collecting together of quantities into one ſum or aggregate (Def. IV.), preſerving their ſigns, and uniting ſuch as can be united (Prop. I.); it is evident, that quantities expreſſed by different ſymbols can be repreſented only AS added together, by being brought into one expreſſion under their ſigns: And like quantities being multiples of ſome ſame quantity (Def. XII.), theſe, when affected with the ſame or like ſigns, are added together *by adding the coefficients, to the ſum adjoining the common quantity, and prefixing the common ſign;* and when affected with unlike ſigns, *by ſubtracting the leſs coefficient from the greater, to the remainder adjoining the common quantity, and prefixing the ſign of the greater.*

165. As every quantity contains itſelf *once* (Def. VI, VII. Corol. I.), any integral quantity, therefore, may be expreſſed as a fraction, by giving to it unity as a denominator; and any fraction may be expreſſed as the numerator of a fraction, having unity for its denominator alſo.

166. And as a fraction is nothing more than an expreſſion of a quotient (Def. VII.), the value of a fraction, therefore, is not affected by any equal multiplication or diviſion of both its terms. And hence the various *reductions* of fractions from this and the former ſimple principles (165).

167. **Fractions** of one and the ſame denomination being like parts of their reſpective wholes or numerators (Def. VII.), are added together *by adding the numerators, and underwriting the common denominator* (Ax. III.). And all fractions may be reduced to equivalent fractions of one and the ſame denomination, by *taking*, according to notation (Def. VI.), *the product of all the denominators for the common denominator, and the product of each numerator into all the denominators but its own for the new numerators*, the terms of the fractions being equally multiplied by this operation (166).

168. *Ratios* are added together as in Propoſition (3.), and according to Definition XVIII. Corollaries I, II. Thus $\overline{A:B}+\overline{C:D}=\overline{\frac{A}{B}:1}+\overline{\frac{C}{D}}:1\times=\frac{A}{B}\times\frac{C}{D}:1=\frac{A}{B}\times\frac{C}{D}=A\times C:B\times D$*.

B.—PROPOSITION II.

169. **Subtraction** determining the difference of quantities (Def. V.), is therefore the converſe of addition (Def. IV.), and takes place abſolutely by *changing the ſign or ſigns of the ablative quantity*, finally abbreviating the general expreſſion of the *difference* by uniting ſuch terms as can be united (Prop. I.).

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* The quantities $\frac{ab}{b}=a$, and $\frac{cd}{d}=$ (Def. VII. Corol. I, II.), may repreſent any two fractions; and the product ac (Def. VI.) is evidently equivalent to $\frac{acbd}{bd}$ (Def. VII.), the product under the numerators and denominators.

170. **Fractions** of one and the ſame denomination are ſubtracted one from another, by ſubtracting the *one numerator from the other, and underwriting the common denominator.*

171. **Ratios** are ſubtracted one from another, as in Propoſition I. and according to Definition XVIII. Corollaries I, II. Thus:

Since by Prop. I. $\overline{A:B}+\overline{C:D}=AC:BD$; therefore, taking away the ratio $C:D$ from each of theſe equals, and the ratio $A:B=\overline{AC:BD}-\overline{C:D}$ (Ax. IV.), or the ratio $\frac{A}{B}:1=\frac{AC}{BD}:1-\frac{C}{D}:1$; and theſe ratios having now the common conſequent, unity, we have only to obſerve what operation muſt obtain among the antecedents $\frac{AC}{BD}$, $\frac{C}{D}$ to bring out the antecedent $\frac{A}{B}$: And here it is manifeſt, that the product of the numerator by D, and of the denominator by C of the antecedent $\frac{AC}{BD}$, gives directly the antecedent $\left(\frac{A\times CD}{B\times CD}=\right)\frac{A}{B}$ (Def. VI. and Corol. I.); but $\frac{D}{C}$ is the inverſe or reciprocal of the antecedent $\frac{C}{D}$ (Def. XI. Corol. I.) Therefore, in taking away or ſubtracting one ratio from another ratio, we muſt *change the ſign of the ablative ratio, and invert the terms thereof*, according to the Propoſition (4).

C.—PROPOSITION III.

172. **Multiplication** being a manifold addition, repeating one quantity ſo often as the ſame or another quantity contains units (Def. VI.). Hence (1.) a ſimple quantity may be multiplied by a ſimple quantity, *by taking the product of the quantities or literal ſymbols, according to notation* (Def. VI.), *and thereto prefixing the product of the coefficients for the coefficient of the product.* (2.) A compound quantity may be multiplied by a ſimple quantity, *by beginning at the left hand, multiplying the ſimple quantity into each term of the multiplicand ſucceſſively*, as in Caſe I. *and making the products* affirmative *or* negative, *according as the factors have* like *or* unlike *ſigns.* And (3.) one compound quantity may be multiplied by another compound quantity, by placing the multiplier under the multiplicand, term under term, that being conſidered as the multiplicand which hath the greater number of terms; then, *beginning at the left hand, each term of the multiplier is drawn or multiplied into each term of the multiplicand ſucceſſively*, as in Caſe II. the different products being placed under one another, with like quantities under like quantities when there are ſuch quantities: And the *whole* product is obtained by collecting together all the different products into one ſum (Prop. I.).

173. **Fractions** are multiplied together, by taking the *product of the numerators for the numerator of the* product, *and the product of the denominators for the denominator of the* product; as demonſtrated, Note A (168). Or the ſame may be demonſtrated in manner following, viz. Suppoſe $\frac{A}{B}=P$, and $\frac{C}{D}=Q$ (Def. VII. Corol. II.), or $A=BP$, and $C=DQ$

(Definition VII. Corol. I.); then $AC = BP \times DQ$ (Ax. IX.) $= PQ \times BD$ (Def. VI.), and conſequently $\frac{AC}{BD} = \frac{PQ \times BD}{BD}$ (Ax. X.) $= PQ$ (Def. VII.) $= P \times Q = \frac{A}{B} \times \frac{C}{D}$ by ſuppoſition.

174. Hence, in multiplying together fractions, the denominators of any of the factors may be interchanged to depreſs the final product. And hence alſo, a fraction may be multiplied by an integral quantity, by multiplying the numerator *by the integral quantity*, or by dividing the *deuominator thereby*.

D.—PROPOSITION IV.

175. Division being a manifold ſubtraction, and the converſe therefore of multiplication, determines how often one quantity may be taken out of, or is contained in, another quantity, by conſidering the dividend as a product, whereof the efficient factors are the diviſor and quotient (Def. VII.). And hence (1.) a ſimple quantity may be divided by a ſimple quantity, by inquiring *what quantity, multiplied into the diviſor, produces the dividend*, or *by placing the dividend above a ſmall line, and the diviſor under it, and* expunging all *numbers and quantities that are common efficients of each*. (2.) A compound quantity may be divided by a ſimple quantity, by beginning at the left hand, dividing as in Caſe I. each term of the dividend by the diviſor, and *making the quotients affirmative or negative, according as the ſigns of the diviſor and dividend are* like *or* unlike. And (3.) one compound quantity may be divided by another compound quantity, by ranging the terms of the diviſor and dividend, according to the dimenſions of any the ſame quantity contained in each, and proceeding as in common arithmetic; the quotient quantities being determined by *dividing the firſt term of the* dividend, *and of every* divdiual, *by the firſt term of the diviſor*.

176. Fractions are divided one by another, by multiplying the *inverſe* or *reciprocal* (Def. XI. Corol. I.) of the fractional diviſor into the fractional dividend, as demonſtrated, Note B (171). Or the ſame may be demonſtrated in manner following, viz. Suppoſe $\frac{A}{B} = P$, and $\frac{C}{D} = Q$ (Def. VII. Corol. II.), or $A = BP$, and $C = DQ$ (Def. VII. Corol. I.), and therefore $AD = BPD$, and $BC = BDQ$ (Ax. IX.); then $\frac{AD}{BC} = \frac{BPD}{BDQ}$ (Ax. X.) $= \frac{P}{Q}$ (Def. VII.) $= P \div Q = \frac{A}{B} \div \frac{C}{D}$ by ſuppoſition.

177. And hence, a fraction may be divided by an integral quantity, by dividing the *numerator by the integral quantity*, or by multiplying the *denominator thereby*.

E.—PROPOSITION VII.

178. Since (Def. XVIII. Corol. I.) the ratio $A : mA = B : mB$ $(= 1 : m)$, and the ratio $mA : A = mB : B$ $(= m : 1)$; where A, mA, B, mB, or mA, A, mB, B, may repreſent *any* four analogous magnitudes; and where m may repreſent *any* number integral, fractional, &c. Therefore, in

any analogy, the firſt term has as much magnitude when compared with the ſecond, as the third when compared with the fourth; and, conſequently, if the firſt term be greater, equal, or leſs than the ſecond, the third term will be *equally* greater, equal, or *equally* leſs than the fourth.

179. And hence all the *theorems* in this Propoſition (13), and Corollaries thereof (14, 15, 16,) as in the following table, viz.

$$A : mA = B : mB \ (= 1 : m)$$

$$\left\{\begin{array}{c} A \times mB = mA \times B \\ A + mA : A = B + mB : B \ (= 1 + m : 1) \\ A + mA : mA = B + mB : mB \ (= 1 + m : 1) \end{array}\right\} \text{Jointly.}$$

$$\left\{\begin{array}{l} A \backsim mA : A = B \backsim mB : B \ (= 1 \backsim m : 1) \\ A \backsim mA : mA = B \backsim mB : mB \ (= 1 \backsim m : m) \end{array}\right\} \text{Disjointly.}$$

$$\left\{\begin{array}{l} A \overset{+}{\backsim} mA : A = B \overset{+}{\backsim} mB : B \ (= 1 \overset{+}{\backsim} m : 1) \\ A \overset{+}{\backsim} mA : mA = B \overset{+}{\backsim} mB : mB \ (= 1 \overset{+}{\backsim} m : m) \end{array}\right\} \text{Converſion.}$$

$$A + mA : A \backsim mA = B + mB : B \backsim mB \ (= 1 + m : 1 \backsim m) \text{ Mixtim.}$$

$$A : B = mA : mB, \text{ Alternation or Permutation.}$$

$$mA : A = mB : B \ (= m : 1) \text{ Inverſion.}$$

$$\left.\begin{array}{l} A : mA \\ B : mB \end{array}\right\} = A + B : mA + mB \ (= 1 : m) \text{ Syllepſis.}$$

$$\left.\begin{array}{l} A : mA \\ B : mB \end{array}\right\} A \backsim B : mA \backsim mB \ (= 1 : m) \text{ Dialepſis.}$$

Hyp. $A : mA = B : mB = C : mC = D : mD$, &c.
Then $A : mA = A + B + C + D$, &c.$: mA + mB + mC + mD$, &c. $(= 1 : m)$.

180. Suppoſe now univerſally, the ratio $A : B = C : D$; then, ſince $\frac{B}{A} \times A = B$ (Def. VIII. Corol. I.), therefore $\frac{B}{A} \times C = D$ (178) the fourth proportional to A, B, C. And hence the common arithmetical rule for finding a fourth proportional, by dividing the product of the ſecond and third by the firſt analogous term.

181. And becauſe $\frac{B}{A} \times C = D$, when the ratio $A : B = C : D$ (180); therefore, by equal multiplication ($B \times C = A \times D$ or) $AD = BC$ (Ax. IX.) as before (178). Or this may be demonſtrated in manner following, viz. Since $A : B = C : D$ (Hyp.), therefore $\overline{A:B} - \overline{C:D} = 0$ (Ax. II.); that is, $\overline{A:B} + \overline{D:C}$, or $AD : BC = 0$ (Prop. II), and conſequently $AD = BC$ (Def. XVIII. Corol. III.). Thus, from an analogy, may be derived an equation between the product of the extremes and means; and converſely, as ſhown in the third Corollary of this Propoſition (16).

182. As the terms of a ratio muſt neceſſarily be of one and the ſame kind or denomination, and as there may be the ſame ratio of magnitude between two quantities or numbers of one kind or denomination, as between

two quantities or numbers of any other kind or denomination (Def. XVIII.): Hence the very general and extenſive application of the doctrine of PROPORTION in all arts, ſciences, buſineſs, &c. And in every caſe of proportion, to determine truly a fourth proportional to three given, the proportionals muſt be ſo ranged, that the two *given* quantities, or numbers of one and the ſame kind, be the terms of the firſt ratio, and the third given quantity, or number of the ſame kind, with that required; the antecedent of the ſecond ratio, the conſequent being the fourth or required term, which may always be repreſented by ſome one of the final letters of the alphabet, x, y, z, v, &c.

183. Then, as things of *greater* value require a *leſs* number, and things of *leſs* value a *greater* number, to make an equal exchange; and as the leſs the number of things the *greater* muſt be their value, and the *greater* the number of things the *leſs* their value, to make them equal. Therefore, conſider from the nature of the caſe propoſed, whether the third term be greater, equal, or leſs than the fourth or required term, x, y, &c. and range the terms of the firſt ratio in the ſame order of magnitude reſpectively; or make the firſt term greater, equal, or leſs than the ſecond term, according as the third term is greater, equal, or leſs than the fourth or required term, x, y, &c. And if any terms are mixed, or conſiſt of different denominations, they may be reduced to the leaſt aſſigned denomination when neceſſary, and the analogy reſtated.

184. The analogy being ſtated as directed (183); if either the firſt and ſecond terms, or the firſt and third terms * be diviſible by any the ſame quantities or numbers, inſtead of theſe terms take the reſpective quotients, and reſtate as often as may be the analogy in terms of the quotients: Then finding the multiplier of the firſt term producing the ſecond (180), and the ſame will be the multiplier of the third term producing the fourth or required term, x, y, &c. (178).

F.—PROPOSITION XV. SCHOLIUM I.

185. In the equation $px - x^2 = rq$, when $x = \frac{1}{2}p$, then $\frac{1}{4}p^2 = rq$; and in every other caſe or value of x, $\frac{1}{4}p^2$ is *greater* than rq. For, ſuppoſe $x = \frac{1}{2}p \pm s$, where $s > s\,\frac{1}{2}p$; then by ſubſtitution $(px - x^2 =)$ $\frac{1}{4}p^2 - s^2 = rq$, and of conſequence $\frac{1}{4}p^2 > r\,rq$. But though the root of the equation $px - x^2 = rq$ in general terms admits of two different interpretations or values, as demonſtrated in this Scholium, and though as in the examples to this caſe of the Propoſition both theſe values may be true, examples, however, may occur in which one only of the values may be taken; and the value to be taken will be determined from the ſtructure of the equation, or from the operation producing the equation. Suppoſe $\sqrt{\overline{\tfrac{1}{2}x}} = 21 - x$, where x is evidently leſs than 21; then (Ax. XI.) $\frac{1}{2}x$ $(= \overline{21 - x}^2)$ $= 441 - 42x + x^2$ (Prop. V.), and hence $42\frac{1}{2}x - x^2 = 441$; where $x = 18$, the only true value.

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* Suppoſe the ratio $A : B = C : D$; then, by adding the ratio $m : 1$ to each, the ratio $mA : B = mC : D$ (Ax. VII.); where m may repreſent any number, integral, fractional, &c.

$$42\tfrac{1}{2}x - x^2 = 441$$

Here $\frac{42\frac{1}{2}}{2} = 21\tfrac{1}{4}$, and $\overline{21\tfrac{1}{4}}^2 = 451\tfrac{9}{16}$

$$\overline{21\tfrac{1}{4}}^2 - \overline{42\tfrac{1}{2}x - x^2}\left(= 451\tfrac{9}{16} - 441\right) = 10\tfrac{9}{16}$$

$$\overline{21\tfrac{1}{4}}^2 - 42\tfrac{1}{2}x + x^2\left(= 10\tfrac{9}{16}\right) = \tfrac{169}{16}$$

$$21\tfrac{1}{4} - x\left(= \tfrac{13}{4}\right) = 3\tfrac{1}{4}$$

$$x\left(= 21\tfrac{1}{4} - 3\tfrac{1}{4}\right) = 18.$$

Or

$$\overline{21\tfrac{1}{4}}^2 - 42\tfrac{1}{2}x + x^2\left(= 10\tfrac{9}{16}\right) = 9 + 1\tfrac{1}{2} + \tfrac{1}{16}$$

$$21\tfrac{1}{4} - x = 3\tfrac{1}{4}$$

$$x = 18.$$

G—PROP. XV. SCHOL. III. or ART. XXX.

186. The theorem here given for evolving a quadratic binomial, may be demonſtrated *ſynthetically*, in manner following, viz.

Since $\overline{B \pm \sqrt{B^2 - Q}}^2 = B^2 \pm 2B\sqrt{B^2 - Q} + B^2 - Q$ (Propoſition V.) $= \overline{2B^2 - Q} \pm \sqrt{4B^2 \times \overline{B^2 - Q}}$ (Prop. IV.), and ſince (Hyp.) $\frac{R+Q}{2} = B^2$, and of conſequence $\left(R - \frac{R+Q}{2} =\right) \frac{R-Q}{2} = B^2 - Q$, as alſo $\sqrt{R^2 - S} = Q$; therefore, by reſtitution, $\overline{B \pm \sqrt{B^2 - Q}}^2 = R \pm \sqrt{R^2 - Q^2} = R \pm \sqrt{R^2 - \overline{R^2 - S}} = R \pm \sqrt{S}$ (Prop. XI.), according to the Theorem (30). Thus: the ſquare of the greater part of the root is $\frac{R+Q}{2}$, or $R + \sqrt{R^2 - S}$; and that of the leſs part of the root $\frac{R-Q}{2}$, or $R - \sqrt{R^2 - S}$.

187. And hence, univerſally, the root of a quadratic binomial may be determined from the ſum and difference of roots of half the ſum and half the difference of the leading terms of the binomial, and the root of the difference of the ſquares of the terms.

Ex. 3. Where $R = 3$, and $S = 8$. Here $Q = 1$, $B = \sqrt{2}$, and $B \pm \sqrt{B^2 - Q} = \sqrt{2} \pm 1$.

Ex. 4. Where $R = 6$, and $S = 20$. Here $Q = 1$, $B = \sqrt{5}$, and $B \pm \sqrt{B^2 - Q} = \sqrt{5} \pm 1$.

And to Example 1. may be added the five following, making *ſix binomials in the tenth book of Euclid.*

Binomial I. $4 + \sqrt{11}$, root of $27\sqrt{704}$.

Binomial II. $\sqrt[4]{12} + \sqrt[4]{\frac{27}{4}}$—$6 + \frac{1}{2}\sqrt{147}$, *former bimedial.*

Binomial III. $\sqrt[4]{\frac{80}{3}} + \sqrt[4]{15}$—$\frac{245}{3} + \sqrt{80}$, *latter bimedial.*

Binomial IV. $\sqrt{\frac{7}{2} + \frac{1}{2}\sqrt{29}} + \sqrt{\frac{7}{2} - \frac{1}{2}\sqrt{29}}$—$7 + \sqrt{20}$, the *major.*

Binomial V. $\sqrt{\sqrt{5} + 1} + \sqrt{\sqrt{5} - 1}$—$7 + \sqrt{40}$, the *potent, a rational with a medial.*

Binomial VI. $\sqrt{\sqrt{5} + \sqrt{3}} + \sqrt{\sqrt{5} - \sqrt{3}}$—$\sqrt{20} + \sqrt{8}$, the *potent two medials.*

H.—PROPOSITION XVI.

188. Since in a right angled plane triangle the tangent of half either angle at the hypothenuſe is to the radius, as the ſide oppoſite to the angle to the hypothenuſe and other ſide taken together: And *inverſely* (14), radius, to the tangent of half either angle at the hypothenuſe, as the ſide oppoſite to the angle to the exceſs of the hypothenuſe above the other ſide (5. e. 2). Hence, to the analytical and geometrical reſolution of quadratic equations in the laſt and in this Propoſition, may be added the *trigonometrical* reſolution of the ſame, as in the following practical rules, viz.

189. RULE I. Increaſe half the logarithm of the abſolute term (rq) by the number 10, and from the ſum ſubtract the logarithm of half the coefficient (p) of the ſecond term; the remainder in Caſes I. and II. is the *logarithmic tangent*, and in Caſe III. the *logarithmic ſine* of an angle A.

RULE II. To and from half the logarithm of the abſolute term (rq), add and ſubtract the *logarithmic tangent* of half the angle A; let the ſum be diminiſhed and the difference increaſed by the number 10, and the numbers reſulting will be the logarithms of the *roots* of the equations in Caſes I. and II.

RULE III. From and to half the logarithm of the abſolute term (rq), ſubtract and add the logarithmic tangent of half the angle A; let the difference be increaſed and the ſum diminiſhed by the number 10, and the numbers reſulting will be the logarithms of the *two roots* of the equation in Caſe III.

Caſe. I. Example $x^2+347x=22110$, to determine x.
Here $p=347$, and $rq=22110$.

The logarithm of rq . . . . $22110 = 4.3445887$

From half thereof . . . . $+10 = 12.1722943$

Take the logarithm of $\frac{1}{2}p$ . . . . $\frac{374}{2} = 2.2392995$

Remains logarithmic tangent of angle A . $40^\circ\ 35'\ 52'' = 9.9329948$

To logarithmic tangent of $\frac{1}{2}$ angle A . . $20^\circ\ 17'\ 56'' = 9.5680714$
Add $\frac{1}{2}$ the logarithm of rq . . . . $22110 = 2.1722943$

The ſum (-10) is the logarithm of x . . . . . . $= 1.7403657$
And $\because x=55$.

Caſe II. Example $x^2-\frac{110}{24}x=\frac{1300}{24}$, to determine x.
Here $p=\frac{110}{24}$, and $rq=\frac{1300}{24}$.

PREPARATION.

From log. of $1300=5.1139434$ From log. of (half $110=$) $55=1.7403627$
Take log. of $24=1.3802112$ Take log. of . . . $24=1.3802112$

Rem. log. $rq\ \frac{1300}{24}=1.7337322$ Remains log. of $\frac{1}{2}p$. $\frac{55}{24}=0.3601515$

From half the logarithm of $rq+10$ . . . $\frac{1300}{24}+10=10.8668661$
Take the logarithm of $\frac{1}{2}p$ . . . . $\frac{55}{24}=0.3601515$

Remains log. tangent of angle A . . . . $72^\circ\ 42'\ 16''=10.5067146$

From half the log. of $rq+10$ . . . . . . $\frac{1300}{24}+10=10.86686661$
Take log. tangent of $\frac{1}{2}$ angle A . . . . $36^\circ\ 21'\ 8''=9.8668617$

Remains the logarithm of . . . . $x=1.0000044$
And $\therefore x=10$.

Caſe III. Example $\frac{264}{25}x-x^2=\frac{695}{25}$, to the values of x.
Here $p=\frac{264}{25}$, and $rq=\frac{695}{25}$.

PREPARATION.

From log of $695=2.8419848$ From log. of . $(\frac{264}{2})131=2.1205739$
Take log. of $25=1.3979400$ Take log. of . . . $25=1.3979400$

Rem. log. $rq\ \frac{695}{25}=1.4440448$ Rem. log. of $\frac{1}{2}p$ . . $\frac{132}{25}=0.7226339$

From half the log. of $rq+10$ . . . . . . $\frac{695}{25}+10=10.7220224$
Take log. of $\frac{1}{2}p$ . . . . $\frac{132}{25}=0.7226339$

Remains log. ſine of angle A . . . . . $86^\circ\ 57'\ 45''=9.9993885$

To and from half the log. rq . . . . $\frac{695}{52}=0.7220224$
Add and ſubtract the log. tang. of $\frac{1}{2}$ angle A $43^\circ\ 28'\ 52''=9.9769646$

The log. of $x=\begin{cases}\text{Sum}-10 \ldots\ldots & =0.6989870\\ \text{Diff.}+10 \ldots\ldots & =0.7450578\end{cases}$
And $\because x=5$, and 5.56 *.

I.—PROPOSITION XVII. AND COROLLARIES.

190. This Propoſition (32) may be demonſtrated *ſynthetically*, in manner following, viz. Suppoſe $y=\sqrt[3]{r+\sqrt{r^2q^3}}$, and $v=\sqrt[3]{r-\sqrt{r^2-q^3}}$; then $z=y+v$ (Hyp.), and $z^3=\overline{y+v}^3$ (Ax. XI.) $=y^3+3y^2v+3yv^2+v^3$ (Prop. V.) $=y^3+3yv\times\overline{y+v}+v^3=y^3+v^3+3yv\times z$ (Hyp.); but, ſince $y=\sqrt[3]{r+\sqrt{r^2-q^3}}$, and $v=\sqrt[3]{r-\sqrt{{}^2-q^3}}$ (Hyp.), therefore $yv=\sqrt[3]{\overline{r+\sqrt{r^2-q^3}}\times\overline{r-\sqrt{r^2-q^3}}}$ (Prop. III.) $=\sqrt[3]{r^2-\overline{r^2-q^3}}=\sqrt[3]{q^3}=q$ (Prop. VI.), and $y^3+v^3=r+\sqrt{r^2-q3}+r-\sqrt{r^2-q3}=2r$ (Pr. I.): Whence, by reſtitution $z^3\ (=y^3+v^3+3yv\times z)=2r+3qz$, and conſequently $z^3-3qz-2r=0$, according to the Propoſition.

In the equation of Corollary II. viz. $z^3+3qz-2r=0$ (34), where $z>s\sqrt[3]{2r}$, and alſo $z>s\frac{2r}{3q}$; and where $r-\sqrt{r^2+q^3}$ is negative, becauſe $\sqrt{r^2+q^3}>r\,r$, as is obvious: Therefore, in this equation

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* See the firſt of DR HALLEY's four Lectures on the Geometrical conſtruction of equations, publiſhed as an Appendix to KERSEY's Algebra; and alſo DODSON's Repoſitory, vol. i. pp. 115, 119, 123.

$$z = \sqrt[3]{r + \sqrt{r^2 + q^3}} - \sqrt[3]{-r + \sqrt{r^2 + q^3}}$$

Or the value of z is as in the other expreſſion of this Corollary, by one evolution of the cubic root.

192. In the equation of Corollary III. viz. $z^3 - 3q\,z + 2r = 0$ (35), or $3qz - z^3 = 2r$; ſince $3qr - z^3 = 2r$, it is evident that $z > s\ \sqrt{3q}$: Suppoſe therefore $z = \sqrt{q}$; then by ſubſtitution, $2q\sqrt{q} = 2r$. Now, as z is always leſs than $\sqrt{3q}$, and may be equal to $\sqrt{q}$; ſo it may fall ſhort of $\sqrt{q}$ by a quantity (s) leſs than $\sqrt{q}$, and it may exceed $\sqrt{q}$ by a quantity (s), which when added to $\sqrt{q}$, ſhall give a ſum leſs than $\sqrt{3q}$; and in both theſe caſes, the quantity $2r$ is leſs than $2q\sqrt{q}$ (or r leſs than $q\sqrt{q}$): For, by ſuppoſing $z = \sqrt{q} \pm s$, we have by ſubſtitution $2q\sqrt{q} - 3\,\overline{\sqrt{q} \pm s} \times s^2 = 2r$, where it is evident that $2q\sqrt{q} > r\ 2r$, or $q\sqrt{q} > r\ r$; and therefore, by equal involution, $q^3 > r\ r^2$.

193. And of the general complete equation of Corollary IV. (36), the third and fourth terms may evidently be repreſented as both affirmative or both negative, or as one affirmative and the other negative.

$$x^3 - 3px^2 - \overline{3q - 3p^2} \times x + \overline{3pq - p^3} - 2r = 0.$$

K.—PROPOSITION XX. AND SCHOLIUM. (49, 50).

194. This Prop. (49) for evolving a cubic binomial, may be demonſtrated in manner following. Since $\overline{B + \sqrt{B^2 - Q}}^{\,3} = 4\,B^3 - 3\,BQ \pm \overline{4B^2 - Q} \times \sqrt{B^2 - Q}$ (Prop. V.) $= 4\,B^3 - 3\,BQ \pm \sqrt{\overline{4B^2 - Q}^{\,2} \times \overline{B^2 - Q}} = 4B^3 - 3BQ \pm \sqrt{16B^6 - 24B^4Q + 9B^2Q^2 - Q^3}$ (Propoſition III.) $= 4\,B^3 - 3\,BQ \pm \sqrt{\overline{4B^3 - 3BQ}^{\,2} - Q^3}$ (Prop. VI.), and ſince by ſuppoſition ($8B^3 - 6BQ = 2\,R$ or) $4\,B^3 - 3\,BQ = R$, as alſo $\sqrt[3]{R^2 - S} = Q$ or $R^2 - S = Q^3$; therefore, by reſtitution, $B \pm \sqrt{B^2 - Q}^{\,3}$ $(= R \pm \sqrt{R^2 - \overline{R^2 - S}}) = R \pm \sqrt{S}$.

195. And that $z^2 + 2mz + 4m^2 - 3q = 0$, as in the Scholium (50), will appear directly on performing the diviſion according to Propoſition IV.; it being conſidered, that as $z = 2m$ (Propoſition XVII.), ſo $8m^3 - 6qm - 2r = 0$.

$$z - 2m)\,z^3 - 3qz - 2r\,(z^2 + 2mz + 4m^2 - 3q$$

$$z^3 - 2mz^2$$

$$2mz^2 - 3qz$$

$$2mz^2 - 4m^2z$$

$$4m^2z - 3qz$$

$$4m^2z - 8m^3$$

$$-3qz + 8m^3$$

$$-3qz + 6qm$$

$$8m^3 - 6qm - 2r\ (= 0).$$

L.—PROPOSITION XXIII. (67).

196. Since by ſuppoſition $A+Bx+Cx^2+Dx^3+\&c.=a+bx+cx^2+dx^3+\&c.$ therefore, by tranſpoſition (19) $\overline{A-a}+\overline{B-b}\times x+\overline{C-c}\times x^2+\overline{D-d}\times x^3+\&c.=0$; if now ($A-a=0$ or) $A=a$, then $\overline{B-b}\times x+\overline{C-c}\times x^2+\overline{D-d}\times x^3+\&c.=0$ *univerſally*, or whatever may be the value of x. But the quantity $\overline{B-b}\times x+\overline{C-c}\times x^2+\overline{D-d}\times x^3+\&c.$ evidently vaniſhes not only when x itſelf vaniſhes, but alſo when the ſeveral coefficients $B-b$, $C-c$, $D-d$, &c. vaniſh: And thus, the ſeries $Bx+Cx^2+Dx^3+\&c.$, $bx+cx^2+dx^3+\&c.$ being mutually convertible, or the ſame, muſt therefore coaleſce into one when $B-b=0$, $C-c=0$, $D-d=0$, &c. and this ſingle ſeries will always converge when x is a converging quantity. And though, in the general equation of this Propoſition, all the terms are repreſented as affirmative, they may be any how affected with the ſigns of addition and ſubtraction.

M.—PROPOSITION XXIV. (70).

197. Since m, n are affirmative integers by ſuppoſition; therefore, by diviſion (Prop. IV. and Corol.), $\frac{v^m-z^m}{v-z}=v^{m-1}+v^{m-2}z+v^{m-3}z^2$ (m), or continued to m terms (A), and in the ſame way $\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2$ (n); whence, ſuppoſing ($x^n=v^m$, $y^n=z^m$ or) $x=v^{\frac{m}{n}}$, $y=z^{\frac{m}{n}}$; then, ſince x^{n-1} $\left(=v^{\frac{m}{n}\times\overline{n-1}}\right)=v^{m-\frac{m}{n}}$, x^{n-2} $\left(=v^{\frac{m}{n}\times\overline{n-2}}\right)=v^{m-\frac{2m}{n}}$, &c. (Propoſition V.); therefore, by ſubſtitution, $\left(\frac{x^n-y^n}{x-y}=\right)\frac{v^m-z^m}{v^{\frac{m}{n}}-z^{\frac{m}{n}}}=$(B) $v^{m-\frac{m}{n}}+v^{m-\frac{2m}{n}}z^{\frac{m}{n}}+v^{m-\frac{3m}{n}}z^{\frac{2m}{n}}$ (n)

But $\left(\frac{v^m-z^m}{v-z}\div\frac{v^m-z^m}{v^{\frac{m}{n}}-z^{\frac{m}{n}}}=\right)\frac{v^{\frac{m}{n}}-z^{\frac{m}{n}}}{v-z}\left(=\frac{A}{B}\right)=$

$$\frac{v^{m-1}+v^{m-2}z+v^{m-3}z^2\ (m)}{v^{m-\frac{m}{n}}+v^{m-\frac{2m}{n}}z^{\frac{m}{n}}+vm-\frac{3-}{n}z^{\frac{2m}{n}}\ (n)}$$

and dividing both ſides of this equation by $v^{\frac{m}{n}-1}$ (Ax. X.) we have $\frac{v^{\frac{m}{n}}-z^{\frac{m}{n}}}{v-z}\times\frac{1}{v^{\frac{m}{n}-1}}=$

$$\frac{v^{m-1}+v^{m-2}z+v^{m-3}z^{2}\ (m)}{v^{m-1}+v^{m-\frac{m}{n}-}z^{\frac{m}{n}}+v^{m-\frac{2m}{n}-1}z^{\frac{2m}{n}}\ (n)}=$$

$$\frac{1+\frac{z}{v}+\overline{\tfrac{z}{v}}\Big|^{2}+\overline{\tfrac{\ }{v}}\Big|^{3}\ (m)}{1+\overline{\tfrac{z}{v}}\Big|^{\frac{m}{n}}+\overline{\tfrac{z}{v}}\Big|^{\frac{2m}{n}}+\overline{\tfrac{z}{v}}\Big|^{\frac{3m}{n}}\ (n)}$$ by dividing both numerator aud denominator by v^{m-1} (Prop. IV.); and hence $\frac{v^{\frac{m}{n}}-z^{\frac{m}{n}}}{v-z}=v^{\frac{m}{n}-1}\times$

$$\frac{1+\frac{z}{v}+\overline{\tfrac{z}{v}}\Big|^{2}+\overline{\tfrac{z}{v}}\Big|^{3}\ (m)}{1+\overline{\tfrac{z}{v}}\Big|^{\frac{m}{n}}+\overline{\tfrac{z}{v}}\Big|^{\frac{2m}{n}}+\overline{\tfrac{z}{v}}\Big|^{\frac{3m}{n}}}$$

N.—PROPOSITION XXV. (80).

198. In the *pure* equation $y^m=N$, ſuppoſe as in the Propoſition $\frac{A}{B}$ to be a near approximation to the value of *y*; and the general *formula* is

$$\frac{N+\overline{m-1}\times\overline{\tfrac{A}{B}}\Big|^{m}}{m\times\overline{\tfrac{A}{B}}\Big|^{m-1}}\quad \text{Or} \quad \frac{NB^{m}+\overline{m-1}\times A^{m}}{mA^{m-1}B}$$

O.—PROPOSITION XXVI. AND COROL. IV. (86, 90).

199. Theſe ſeries, for the logarithms of $\frac{1}{1-n}$, and $1+n$, may be derived in the moſt ſimple and obvious manner from the following obſervations, viz. (1.) In every ſyſtem of logarithmic numbers, the logarithm of *unity* is nothing. (2.) The ſums and differences of any number of logarithmic quantities, are themſelves logarithmic quantities. (3.) The product P of any quantity A by a logarithmic quantity L, is always a logarithmic quantity; for ſince A × L = P (Hyp.), the ratio 1 : A = L : P (Prop. VII. Cor. III.); and L being a logarithmic quantity by ſuppoſition, it is evident that the *analogous* quantity P is of the ſame kind (Def. IV. e. 5). And hence, (4.) The quantity A × L being always a logarithmic quantity, it may therefore be aſſumed as *a* logarithmic of A; wherefore, reſolving the quantity A by diviſion (Prop. IV.), into its conſtituent parts, $\underline{a+b+c+d+e+}$, &c. and ſuppoſing K, M, N, P, Q, &c. to be logarithmic quantities, giving K*a*, M*b*, N*c*, P*d*, Q*e*, &c. logarithms of *a*, *b*, *c*, *d*, *e*, &c. reſpectively, we ſhall have this general expreſſion in a ſeries for defining the logarithm of any quantity A, viz.

$Ka \pm Mb \pm Nc \pm Pd \pm Qe \pm$ &c.

200. And hence, Caſe I. To determine the ſeries for the logarithm of $\frac{1}{1-n}$, or for the meaſure of the *ratio* $\frac{1}{1-n} : 1$.

Here $A \left(=\frac{1}{1-n}\right) = 1+n+n^2+n^3+n^4+$&c. (Prop. IV.) $= a+b+c+d+e+$ &c. by ſuppoſition; whence, ſince $a=1$, therefore $K=0$ (Obſ. 1.), and of conſequence $L, \frac{1}{1-n} = Mn+Nn^2+Pn^3+Qn^4+$&c. or $2L, \frac{1}{1-n} = 2Mn+2Nn^2+2Pn^3+2Qn^4+$ &c. But by the principles of logarithms, $2L, \frac{1}{1-n} = L, \left.\overline{\frac{1}{1-n}}\right|^2 = L, \frac{1}{1-\overline{2n-n^2}}$ (Prop. V. Cor. II.), therefore, in the ſeries for the $L, \frac{1}{1-n}$, if the quantity $2n-n^2$ be ſubſtituted inſtead of n, we ſhall have $L, \left.\overline{\frac{1}{1-n}}\right|^2$ $(= M\times\overline{2n-n^2}+N\times\overline{2n-n^2}^2+P\times\overline{2n-n^2}^3+Q\times\times\overline{2n-n^2}^4+$&c.$) = 2M\times n+4\,\overline{N-M}\times n^2+\overline{8P-4N}\times n^3+\overline{16Q-12P+N}\times n^4+$&c. by expanding the terms $= 2M\times n+2N\times n^2+2P\times n^3+2Q\times n^4+$ &c. as above, and of conſequence (Prop. XXIII.) $N=\frac{1}{2}M$, $P=\frac{1}{3}M$, $Q=\frac{1}{4}M$, &c. (67): Thus, $2L, \frac{1}{1-n} = 2Mn+Mn^2+\frac{2}{3}Mn^3+\frac{2}{4}Mn^4+$&c. and therefore $L\frac{1}{1-n} = M\times\overline{n+\frac{1}{2}n^2+\frac{1}{3}n^3+\frac{1}{4}n^4+}$ &c.

201. Caſe II. To determine the ſeries for the logarithm of $1+n$, or for the meaſure of the *ratio* $1+n : 1$.

Since the ratio propoſed is $1+n : 1$, its reciprocal or inverſe will be $1 : 1+n$ (Prop. VII. Corol. I.) or $\frac{1}{1+n} : 1$; therefore, ſince $\frac{1}{1+n} = 1-n+n^2-n^3+n^4$ &c. (Propoſition IV.) proceeding as in Caſe I. we ſhall have $\left(L, \left.\overline{\frac{1}{1+n}}\right|^2 = 2L, \frac{1}{1+n} =\right) -2M\times n+\overline{4N-M}\times n^2-\overline{8P-4N}\times n^3+\overline{16Q-12P+N}\times 4^4$, &c. $= -2M\times n+2N\times n^2-2P\times n^3+2Q\times n^4$, &c. and hence $L, \frac{1}{1+n} = -M\times\overline{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}4^4}$, &c. or $L, 1+n = M\times\overline{n-\frac{1}{2}n^3+\frac{1}{3}n^3-\frac{1}{4}n^4}$, &c. (Prop. II.)

202. And as by the ſeries in Corollary IV. (90), the number $1\mp n$ may be determined from its logarithm L being given, and that whatever may be the relation of magnitude or quantity between L the logarithm and M the *modulus* of the ſyſtem; if, therefore, it be conſidered that equality, ſameneſs, or identity, is the moſt ſimple relation that may obtain between the quantities L, M (Ax. I.), upon this ſuppoſition the ratio $1+n : 1 = 2+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+$, &c. $: 1 = 2.7182818459$, &c. $: 1$; or the ratio $1 : 1-n = 1 : \frac{1}{2}-\frac{1}{2.3}+\frac{1}{2.3.4}-$ &c. $= 1 : 0.367879441171$, &c.;

&c. : And either of theſe numeral ratios is that called by the learned Mr Cotes, the *ratio modularis ;* the *modulus* or module being always the meaſure of this ratio in every logarithmic ſyſtem.

P.—PROP. XXVIII. COROL. II. ART. 122, 124, 125, and 126.

203. The general ſimple intereſt equation $\overline{tr+1}\times p=s$ (122), where r is the intereſt of L 1. for 1 year, called the *ratio of the rate*, as being always the 100th part of the rate *per cent. per annum*, applies in every inſtance of any ſum of money paid *after* or *before* due. In the firſt caſe p is the principal and s the amount thereof, and its intereſt trp for time t and ratio of rate r. In the ſecond caſe, ſuppoſing p, s to change places, the equation becomes $\overline{tr+1}\times s=p$: And here, s is the *preſent worth* or *value* of the principal p for time t and ratio of rate r; or s is the principal which in time t and at ratio of rate r would amount to the ſum or principal p. And as the exceſs of the amount above the principal is the *intereſt ;* ſo the exceſs of the principal above the preſent worth is the *rebate* or *diſcount*.

204. If, therefore, the intereſt of any given principal for any given time and rate, expreſſed in pounds and decimal places when there are places, be divided by the yearly amount of L 1. for the ſame rate, the quotient will be the rebate of ſaid principal for the given time and rate. And if the *yearly* intereſt of any principal, at any rate *per cent.*, be divided by the yearly amount of L 1. at the ſame rate, the quotient will be the yearly rebate of ſaid principal.

205. In the *third* and *fourth* lines of the table (124), the values of r, t are wrong printed, and ſhould have been expreſſed in manner following, viz.

s, t, a	$r=\dfrac{\frac{s}{t}-a}{\frac{t-1}{2}a-s}$
s, r, a	$t=\sqrt{\dfrac{2s}{ar}-\overline{\dfrac{1}{r}-\dfrac{s}{a}+\dfrac{1}{2}}^{2}}-\overline{\dfrac{1}{r}-\dfrac{s}{a}+\dfrac{1}{2}}$

206. The method of computing *accurately* the intereſt of any ſum or principal for any number of days not greater than (a year or) 365, as given in article 125, being *general* for any whatever rate of intereſt, and not confined or limited, as there ſtated, to the rate of 5 per cent. ; the general calculation, therefore, of intereſt for any time or number of days not greater than a year, and at any whatever rate *per cent. per annum*, may be performed with great facility in manner following, viz.

207. *Let the product under the ſum or principal, the number of days*, and the double of the rate per cent. *be advanced five places to the right ; multiply this number by* 3, *advancing the product one place to the right : Subtract the latter advanced product from the former, advancing the remainder one place to the right ; and the ſum of the two advanced products and advanced remainder, diminiſhed by itſelf when advanced four places to the right, will be the true intereſt in every caſe.*

The

208. The example for illuſtrating the general ſimple intereſt table (126) being quite perverted by the *Printer*, is here given properly corrected as follows, viz. Taking the former example (125).

4239.767 × 5 × 256 = 54269|01.76

Then 50000	L 136	19	8	2.85
4000	10	19	2	0.55
200	0	10	11	2.30
60	0	3	3	1.81
9	0	0	5	3.67
Intereſt required,	L 148	13	7	3.18

209. And in uſing this general table (126) to the greateſt advantage, ſuppoſing two additional columns on the right hand as below; *let the product of principal rate and time*, when divided by 100, *be reſolved into its component parts, and the correſponding intereſt, numbers, taken from the table, and added together, will give in the ſum total, the intereſt required.*

Two additional columns to the general table (126).

No.	Q.	No.	Q.
1	2.63	.05	.13
.9	2.37	.04	.11
.8	2.10	.03	.08
.7	1.84	.02	.05
.6	1.58	.01	.026
.5	1.32	.009	.004
.4	1.05	008.	.002
.3	.75	.007	.018
.2	.53	.006	.015
.1	.26	.005	.013
.09	.24	.004	.012
.08	.21	.003	.009
.07	.18	.002	.005
.06	.16	.001	.003

210. But the following, perhaps, is the beſt general and perfectly accurate method of computing the intereſt of money for any number of days not greater than 365, at any whatever rate *per cent. per annum*, and equally eaſy with the common method at the rate of 5 *per cent.* only, viz.

Divide the product of the ſum or principal, the number of days, and the double of the rate, by 73000, *and the quotient will be the intereſt accurately in every caſe.*

211. And hence, to determine the rebate or diſcount at any rate of intereſt, of any ſum or principal paid any number of days not greater than 365 before due; *divide the product of the ſum or principal, the number of days*, and the double of the rate, *by* 73000 *increaſed by the product of the number of days* and double of the rate, *and the quotient will be the* rebate *accurately in every caſe.*

Q.—PROP. XXIX. COROL. XIII. XV. ART. 140, and 147.

212. The EUCLIDEAN ſeries determining all PERFECT NUMBERS, as given in Corollary XIII. (140) being quite perverted by the *Printer*, is here reprinted accurately as follows, viz.

$s=1$	$1+2.\overline{2-1}$	$1+2.\overline{2^2-1}$	$1+2.\overline{2^4-1}$	$1+2.\overline{2^6-1}$	$1+2.\overline{2^8\ 1}$	$1+2.\overline{2^{10}-1}$
$2^{n-1}=1$	2	2^2	2^4	2^6	2^8	2^{10} &c.
$n=1$	2	3	5	7	9	11 &c.

213. And the latter part of article 147, which ſhould here come in after the colon (:) in the 6th line, having been omitted by the *Printer*, is here added as containing an uſeful Theorem, in the following terms, viz.

In the following tables, if N repreſent the tabular number corresponding to any age, the value of the annuity, &c. a will be $a\times N$ (146); and ſubtracting this from the perpetuity $\frac{a}{r}$ (145), the value of the *reverſion* of a fee-ſimple eſtate, &c. (a), after a ſingle life of a given age will be had, viz. $a\times\overline{\frac{1}{r}-N}$.

FINIS.

ERRATA.

Article 168, line 1, read in *the* Propoſition.
——————— — 1, Note (*) read $\frac{cd}{d}=c$
——- 171, — 1, read *Propoſition II.*
——- 181, — laſt, read *ſhown.*
——- 187, — 2, read of *the* roots.
——- 189, — 1, read to *determine* the values of x.

SYNOPSIS

OF

BOOK V.

OF

EUCLID'S ELEMENTS,

ACCORDING TO

DR SIMSON'S EDITION.

ST ANDREWS:

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M.DCC.XCVII.

PREFACE.

THIS SYNOPSIS, firſt drawn up according to Keil's edition of Commendine's Euclid, was afterwards changed, where neceſſary, according to Dr Simſon's edition of the Elements, and is now publiſhed with the addition of a very ſimple demonſtration of Propoſition VIII. from the learned Mr Playfair's Elements of Geometry, for the uſe of Students, who too generally neglect the truly elegant demonſtrations of Euclid, in this moſt uſeful part of the Elements, and thereby obſtruct their improvement in the other branches of Mathematics; and in Natural Philoſophy, very conſiderably.

DEFINITIONS.

DEFINITION I.

A leſs magnitude is ſaid to be a part of a greater magnitude, when the leſs meaſures the greater; that is, when the leſs is contained a certain number of times exactly in the greater.

Thus A is a part of *m* A, as being contained *m* times exactly in *m* A, *m* being a whole number.

DEFINITION II.

A greater magnitude is ſaid to be a multiple of a leſs, when the greater is meaſured by the leſs; that is, when the greater contains the leſs a certain number of times exactly.

Thus *m* A is a multiple of A by *m*; becauſe *m* A contains A *m* times exactly, *m* being a whole number.

DEFINITION III.

Ratio is a certain mutual habitude, or compariſon of two magnitudes of the ſame kind, as to quantity or magnitude.

SCHOLIUM.

A ratio is expreſſed by interpoſing between the magnitudes compared together, or the terms of the ratio, two points placed vertically (:) *; the former term of the ratio being called the* antecedent, *and the latter the* conſequent. *Thus, the expreſſion A : B denotes the ratio of A to B, or the abſolute relation as to magnitude of the antecedent A to the conſequent B, or of the conſequent B to the antecedent A, and that whether this relation be aſſignable or not.*

When the antecedent of a ratio exceeds the conſequent, the ratio is ſaid to be a ratio of greater inequality *or of* majority; *when the antecedent of a ratio is exceeded by the conſequent, the ratio is ſaid to be a ratio of* leſs inequality *or of* minority; *and when the antecedent of a ratio neither exceeds nor is exceeded by the conſequent, the ratio is ſaid to be a ratio of* equal *to* equal *or of* equality.

DEFINITION IV.

Magnitudes are ſaid to have a ratio to one another, when the leſs can be multiplied ſo as to exceed the other.

COROLLARY I.

As of two magnitudes the leſs cannot, by any multiplication whatever, exceed the other, unleſs the magnitudes are of one and the ſame kind, or homogeneous; ſo, between heterogeneous magnitudes there can be no ratio.

COROLLARY II.

As of two magnitudes there can be no diſtinction of leſs or greater, unleſs the magnitudes be unequal; ſo, between equal magnitudes there can be no ratio properly ſo called: A ratio of equality, therefore, is ſaid to be nothing, or equal to nothing; and converſely, of a ratio equal to nothing, the terms are equal.

DEFINITION V.

The firſt of four magnitudes is ſaid to have the ſame ratio to the ſecond which the third has to the fourth, when any equimultiples whatſoever of the firſt and third being taken, and any equimultiples whatſoever of the ſecond and fourth. If the multiple of the firſt be leſs than that of the ſecond, the multiple of the third is alſo leſs than that of the fourth; or if the multiple of the firſt be equal to that of the ſecond, the multiple of the third is alſo equal to that of the fourth; or if the multiple of the firſt be greater than that of the ſecond, the multiple of the third is alſo greater than that of the fourth.

Let A, B, C, D, be four magnitudes, and *m*, *n*, any whatever numbers.

$$\text{A, B, C, D,}\qquad \left.\begin{array}{c}A\\C\end{array}\right\} m = \left\{\begin{array}{l}mA\\mC\end{array}\right. \qquad \left.\begin{array}{c}B\\D\end{array}\right\} n = \left\{\begin{array}{l}nB\\nD\end{array}\right.$$

$$\text{A, B, C, D,}\left\}\begin{array}{l}\text{If } mA > r = \text{or} > s\ nB\\ \text{and } mC > r = \text{or} > s\ nD\end{array}\right\{ \text{The ratio } A : B = C : D$$

Thus, of the four numbers 3, 2, 81, and 54, the ratio $3 : 2 = 81 : 54$.

$$\left.\begin{array}{r}3\\81\end{array}\right\} 5 \qquad \begin{array}{r}15\\405\end{array}\Bigg| \qquad \left.\begin{array}{r}2\\54\end{array}\right\} 7 \qquad \begin{array}{r}14\\378\end{array}$$

Here $15 > r\ 14$, and $405 > r\ 378$

$$\left.\begin{array}{r}3\\81\end{array}\right\} 6 \qquad \begin{array}{r}18\\486\end{array}\Bigg| \qquad \left.\begin{array}{r}2\\54\end{array}\right\} 9 \qquad \begin{array}{r}18\\486\end{array}$$

Here $3 \times 6 = 2 \times 9 (= 18)$, and $81 \times 6 = 54 \times (9 = 486)$

$$\left.\begin{array}{r}3\\81\end{array}\right\} 8 \qquad \begin{array}{r}24\\648\end{array}\Bigg| \qquad \left.\begin{array}{r}2\\54\end{array}\right\} 14 \qquad \begin{array}{r}28\\756\end{array}$$

Here $24 > s\ 28$, and $648 > s\ 756$

And the ſame will appear according to any multiplication whatever. And thus alſo, of the four numbers 5, 7, 115, and 161, the ratio $5 : 6 = 115 : 161$.

$$\left.\begin{matrix}5\\115\end{matrix}\right\}10 \qquad \begin{matrix}50\\1150\end{matrix}\;\Bigg| \qquad \left.\begin{matrix}7\\161\end{matrix}\right\}6 \qquad \begin{matrix}42\\966\end{matrix}$$

Here $50 > r\ 42$, and $1150 > r\ 966$

$$\left.\begin{matrix}5\\115\end{matrix}\right\}7 \qquad \begin{matrix}35\\805\end{matrix}\;\Bigg| \qquad \left.\begin{matrix}7\\161\end{matrix}\right\}5 \qquad \begin{matrix}35\\805\end{matrix}$$

Here $5\times 7=7\times 5(=35)$, and $15\times 7=161\times 5(=805)$

$$\left.\begin{matrix}5\\115\end{matrix}\right\}10 \qquad \begin{matrix}50\\1150\end{matrix}\;\Bigg|\;\begin{matrix}7\\161\end{matrix} \qquad \Big\}12 \qquad \begin{matrix}84\\1932\end{matrix}$$

Here $50 > s\ 84$, and $1150 > s\ 1932$

And the ſame will appear according to any multiplication whatever.

SCHOLIUM.

To compare together as to equality, in the eaſieſt manner, numeral ratios, or ratios expreſſed in numeral terms; let them be reduced to equivalent ratios in one and the ſame ſeries, by taking equimultiples of the terms of each ratio by the alternate conſequents or antecedents: And if the ratios thus reduced to, or brought to be in one and the ſame ſeries, be expreſſed by the ſame terms, the ratios compared together are equal to one another.

Ratios. Reduced to the ſame ſeries. Equal.

$$\begin{matrix}3:2\\81:54\end{matrix}\left\{\begin{matrix}3:2=3\times 54:2\times 54=162:108\\81:54=81\times 2:54\times 2=162:108\end{matrix}\right\}3:2=81:54$$

Ratios. Reduced to the ſame ſeries. Equal.

$$\begin{matrix}5:7\\115:161\end{matrix}\left\{\begin{matrix}5:7=5\times 115:7\times 115=575:805\\115:161=115\times 5:161\times 5=575:805\end{matrix}\right\}5:7=115:161$$

DEFINITIONS VI, VIII, XII.

Magnitudes which have the ſame ratio are called proportionals; and the terms of two ratios are ſaid to compoſe an *analogy*, which is uſually expreſſed by ſaying, the firſt magnitude is to the ſecond as the third is to the fourth. And in an analogy the two antecedents are called *homologous* to one another, as alſo the two conſequents; but the terms of the ſame ratio are called *analagous* to one another.

DEFINITION VII.

When of the equimultiples of four magnitudes (taken as in Def. V.) the multiple of the firſt is greater than that of the ſecond, but the multiple of the third is not greater than the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third magnitude has to the fourth; and, on the contrary, the third is ſaid to have to the fourth a leſs ratio than the firſt has to the ſecond.

$$\begin{array}{cccc} & A:B,\ C:D & & \mathit{Then} \\ \left.\begin{matrix}A\\C\end{matrix}\right\}m & \left.\begin{matrix}B\\D\end{matrix}\right\{n & \left\{\begin{matrix} mA > r\ nB \\ mC\ \mathit{nct} > r\ nD\end{matrix}\right\} & \begin{matrix}A:B > r\ C:D\\ c:D > s\ A:B\end{matrix}\end{array}$$

SCHOLIUM.

In this definition for comparing together unequal ratios, it is aſſumed by Euclid, *that a ratio of majority is greater the* farther *it is removed from, and a ratio of minority greater the* nearer *it is to, the ratio of equality: Therefore, numeral ratios, or ratios expreſſed in numeral terms, of the ſame kind, and belonging to the ſame ſeries, may be compared together as to greater or leſs, by conſidering only their diſtances from the ratio of equality; a ratio of majority being greater the* farther *it is removed from, and a ratio of minority greater the* nearer *it is removed to, the ratio of equality.*

$$1:5,\ 2:5,\ 3:5,\ 4:5,\ \overline{\underline{5:5}},\ 6:5,\ 7:5,\ 8:5,\ 9:5,\ \&c.$$
$$\&c.\ 5:9,\ 6:9,\ 7:9,\ 8:9,\ \underline{9:9},\ 10:9,\ 11:9,\ 12:9,\ 13:9,\ \&c.$$

In theſe ſeries, the ratios on the right hand of the limit or ratio of equality, are ratios of majority regularly increaſing, *and thoſe on the left hand ratios of minority regularly* decreaſing, *according to* Euclid.

When ratios are not in the ſame ſeries, they may be reduced to one and the ſame ſeries, and compared together as above, by taking equimultiples of the terms of each ratio by the alternate conſequents or antecedents (Def. V. Scholium).

$$\left.\begin{array}{c} \mathit{Unequal\ ratios.} \\ 7:5,\ 11:9 \\ 7:5 = 7\times9:5\times9 = 63:45 \\ 11:9 = 11\times5:9\times5 = 55:45 \\ \underline{\mathit{And} \because 7:5 > r\ 11:9} \\ \underline{45:45}\ldots\ldots 55:45 \ldots\ldots 63:45 \end{array}\right\} \quad \left\{\begin{array}{c} \mathit{Unequal\ ratios.} \\ 5:7,\ 9:11 \\ 5:7 = 5\times11:7\times11 = 55:77 \\ 9:11 = 9\times7:11\times7 = 63:77 \\ \underline{\mathit{And} \because 5:7 > s\ 9:11} \\ \underline{77:77}\ldots\ldots 63:77 \ldots\ldots 55:77 \end{array}\right.$$

And in comparing together unequal ratios, according to this definition of Euclid, *the multiples to be taken are the terms of the leſs ratio* inverted, *or any numbers continually approximating to, and finally terminating in, theſe terms.*

Unequal ratios, 7 : 5, 11 : 9

$$\begin{array}{lcccr} 9)11(1 & & & & 1,\quad 4,\quad 2 \\ \quad\underline{9} & & & & 1:1 \\ \quad 2)9(4 & \left.\begin{matrix}1\\1\end{matrix}\right\}\times4 & + & \left.\begin{matrix}0\\1\end{matrix}\right\} & = 4:5 \\ \qquad\underline{8} & & & & \\ \qquad 1)2(2 & \left.\begin{matrix}4\\5\end{matrix}\right\}\times2 & + & \left.\begin{matrix}1\\1\end{matrix}\right\} & = 9:11 \\ \qquad\quad 2 & & & & \end{array}$$

$$\left.\begin{matrix}7\\11\end{matrix}\right\}9 \quad \begin{matrix}63\\99\end{matrix} \quad \left.\begin{matrix}5\\9\end{matrix}\right\}11 \quad \begin{matrix}55\\99\end{matrix} \quad \left\{\begin{matrix}\mathit{Here}\ 63 > r\ 55\\ \mathit{And}\ 99 = \mathit{not} > r\ 99\end{matrix}\right.$$
$$\mathit{And} \because 7:5 > r\ 11:9$$

Unequal ratios, 7 : 5, 11 : 9

$$\left.\begin{matrix}7\\11\end{matrix}\right\}4\qquad\left.\begin{matrix}28 & 5\\44 & 9\end{matrix}\right\}5\qquad\begin{matrix}25\\45\end{matrix}\qquad\left\{\begin{matrix}\textit{Here } 28 > r\ 25\\\textit{And } 44 > s\ \textit{not} > r\ 45\end{matrix}\right.$$

And ∵ 7 : 5 > *r* 11 : 9

Unequal ratios, 9 : 11, 5 : 7

$$\left.\begin{matrix}9\\5\end{matrix}\right\}7\qquad\left.\begin{matrix}63 & 11\\35 & 7\end{matrix}\right\}5\qquad\begin{matrix}55\\35\end{matrix}\qquad\left\{\begin{matrix}\textit{Here } 63 > r\ 55\\\textit{And } 35 = \textit{not} > r\ 35\end{matrix}\right.$$

And ∵ 9 : 11 > *r* 5 : 7 *.

DEFINITION IX.

Proportion or analogy conſiſts of three terms at leaſt, in which caſe the ſame magnitude is a conſequent of one ratio, and an antecedent of the other; and ſuch ratios are called *continued* ratios; or three ſuch magnitudes are called *continued* or *continual* proportionals.

DEFINITION X, XI.

When three magnitudes are proportional, the firſt is ſaid to have to the *third* the duplicate ratio of that which it has to the *ſecond;* and when four magnitudes are in continued proportion, the *firſt* is ſaid to have to the *fourth* the triplicate ratio of that which it has to the *ſecond;* and ſo on quadruplicate, &c. increaſing the denomination ſtill by unity, in any number of proportionals.

That is, in any number of continued equal ratios; the ratio of the firſt antecedent and laſt conſequent is *multiplicate* of the firſt ratio (or of any one of the equal ratios), by the *number* denoting the number of equal ratios in the ſeries.

Suppoſe the ratio A : B = B : C = C : D = D : E = E : F, &c. Then when A : B = B : C, the ratio A : C is ſaid to be to the duplicate of the ratio A : B. When A : B = B : C = C : D, the ratio A : D is ſaid to be triplicate of the ratio A : B. When A : B = B : C = C : D = D : E, the ratio A : E is ſaid to be quadruplicate of the ratio A : B. When A : B = B : C = C : D = D : E = E : F, the ratio A : F is ſaid to be quintuplicate of the ratio A : B, &c. &c.

This definition is the addition *of continued equal ratios.*

DEFINITION A.

When there are any number of magnitudes of the ſame kind, the firſt is ſaid to have to the laſt of them the ratio compounded of the ratio which the firſt has to the ſecond, and of the ratio which the ſecond has to the third, and of the ratio which the third has to the fourth; and ſo on unto the laſt magnitude.

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* *Vid. Analyſis, Def. XVIII. Corol. V. Scholium.*

For example, if A, B, C, D, be four magnitudes of the ſame kind, the firſt, A, is ſaid to have to the laſt, D, the ratio compounded of the ratio A to B, and of the ratio B to C, and of the ratio C to D; or, the ratio of A to D is ſaid to be compounded of the ratios A to B, B to C, and C to D. And if A has to B the ſame ratio which E has to F, and B to C the ſame ratio that G has to H, and C to D the ſame ratio that K has to L; then by this definition A is ſaid to have to D the ratio compounded of ratios, which are the ſame with the ratios of E to F, G to H, and K to L. And the ſame thing is to be underſtood when it is more briefly expreſſed, by ſaying A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the ſame things being ſuppoſed, if M has to N the ſame ratio which A has to D; then, for ſhortneſs' ſake, M is ſaid to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

This definition is the addition *of continued unequal ratios.*

DEFINITION XIII, XIV, XV, XVI, XVII.

When the middle terms of an analogy change places, this is called *alternation* or *permutation.* When the terms of a ratio change places, the ratio is ſaid to be *inverted.* And when the ſum or difference of both terms of a ratio is compared with either term, the compariſon is ſaid to be *jointly* or *disjointly:* But when either term of a ratio is compared with the difference of the terms, which ever is the greater, this compariſon is called *converſion.*

Let there be any ratio, as $A : B$.

The ratio $A : B$, taken by inverſion, is $B : A$.

The ratio $A : B$, taken jointly, is $A + B : A$ or $A + B : B$.

The ratio $A : B$, taken disjointly, is $A \backsim B : A$ or $A \backsim B : B$.

The ratio $A : B$, taken by converſion, is $A : A \backsim B$ or $B : A \backsim B$.

DEFINITION XVIII, XIX.

When there is any number of magnitudes more than two in one ſeries, and as many others in other ſeries, which are proportional when taken regularly two and two in each ſeries, from the firſt to the laſt, that is, when the firſt magnitude is to the ſecond of the firſt ſeries as the firſt to the ſecond of the other ſeries; and as the ſecond is to the third of the firſt ſeries, ſo is the ſecond to the third of the other ſeries; and ſo on in order: Then are the magnitudes ſaid to be *ordinately* proportional, or in *ordinate* proportion. And when it is inferred, that the firſt is to the laſt of the firſt ſeries of magnitudes as the firſt to the laſt of the others, this inference is ſaid to be *ex aequali*, or *ex aequo*, from equality.

The inference in this definition is the addition *of continued ratios.*

Let there be a ſeries of magnitudes, A, B, C, D, E, F, &c. and another ſeries of magnitudes, as a, b, c, d, e, f, &c.

And ſuppoſe the ratio $A : B = a : b$
$B : C = b : c$
$C : D = c : d$
$D : E = d : e$
$E : F = e : f$
&c. &c.
Then, by equality, the ratio $A : F = a : f$.

DEFINITION XX.

When there is any number of magnitudes more than two in one, and as many others in other ſeries, which are proportional when taken two and two in each ſeries in the following order, viz. When the firſt magnitude is to the ſecond of the firſt ſeries, as the laſt but one is to the laſt of the other ſeries; and as the ſecond is to the third of the firſt ſeries, ſo is the laſt but two to the laſt but one of the other ſeries; and as the third is to the fourth of the firſt ſeries, ſo is the third from the laſt to the laſt but two of the other ſeries; and ſo on in a croſs order: Then are thoſe magnitudes ſaid to be *inordinately* proportional. And when it is inferred that the firſt is to the laſt of the firſt ſeries of magnitudes, as the firſt to the laſt of the others, this inference is ſaid to be *ex aequali, in proportione perturbata, ſeu inordinata*, from equality, in perturbate or diſorderly proportion.

The inference in this definition is the addition *of continued ratios.*

Let there be a ſeries of magnitudes, A, B, C, D, E, F, &c. and another ſeries of magnitudes, a, b, c, d, e, f, &c.
And ſuppoſe the ratio $A : B = e : f$
$B : C = d : e$
$C : D = c : d$
$D : E = b : c$
$E : F = a : b$
Then, by perturbate equality, the ratio $A : F = a : f$.

AXIOMS.

AXIOM I.

Equimultiples of the ſame, or of equal magnitudes, are equal to one another.

AXIOM II.

Thoſe magnitudes of which the ſame or equal magnitudes are equimultiples, are equal to one another.

AXIOM III.

A multiple of a greater magnitude is greater than the ſame multiple of a leſs.

AXIOM IV.

That magnitude of which a multiple is greater than the ſame multiple of another, is greater than that other magnitude.

PROPOSITIONS.

PROPOSITION I.—THEOREM.

If any number of magnitudes be equimultiples of as many, each of each, what multiple ſoever any one of them is of its part, the ſame multiple ſhall all the firſt magnitudes be of all the other.

$$\begin{array}{c|c|l} m\,A & A & m \times A \\ m\,B & B & m \times B \\ m\,C & C & m \times C \\ m\,D & D & m \times D \\ \&c. & \&c. & \&c. \end{array}$$

$$m\,A + m\,B + m\,C + m\,D,\ \&c. = m \times \overline{A + B + C + D,\ \&c.}$$

PROPOSITION II.—THEOREM.

If the firſt magnitude be the ſame multiple of the ſecond that the third is of the fourth, and the fifth the ſame multiple of the ſecond that the ſixth is of the fourth; then ſhall the firſt together with the fifth be the ſame multiple of the ſecond, that the third together with the ſixth is of the fourth.

$$m\,A,\ A,\ m\,B,\ B,\ n\,A,\ n\,B$$

$$m\,A + n\,A = \overline{m + n} \times A$$

$$m\,B + n\,B = \overline{m + n} \times B.$$

COROLLARY.

From this it is plain, that if any number of magnitudes, $m\,A$, $n\,A$, $p\,A$, be multiples of another A, and as many $m\,B$, $n\,B$, $p\,B$, be the ſame multiples of B, each of each, the whole of the firſt is the ſame multiple of A that the whole of the laſt is of B.

$$m\,A,\ n\,A,\ p\,A$$

$$m\,B,\ n\,B,\ p\,B$$

$$m\,A + n\,A + p\,A = \overline{m + n + p} \times A$$

$$m\,B + n\,B + p\,B = \overline{m + n + p} \times B.$$

PROPOSITION III.—THEOREM.

If the firſt be the ſame multiple of the ſecond which the third is of the fourth; and if of the firſt and third there be taken equimultiples, theſe ſhall be equimultiples, the one of the ſecond, and the other of the fourth.

Hyp.

$$\left.\begin{array}{l} m\,\mathrm{A},\ \mathrm{A},\ m\,\mathrm{B},\ \mathrm{B}, \\ n\,m\,\mathrm{A},\ n\,m\,\mathrm{B}, \end{array}\right\} \begin{array}{l} n\,m\,\mathrm{A} = n\,m \times \mathrm{A} \\ n\,m\,\mathrm{B} = n\,m \times \mathrm{B}. \end{array}$$

PROPOSITION IV.—THEOREM.

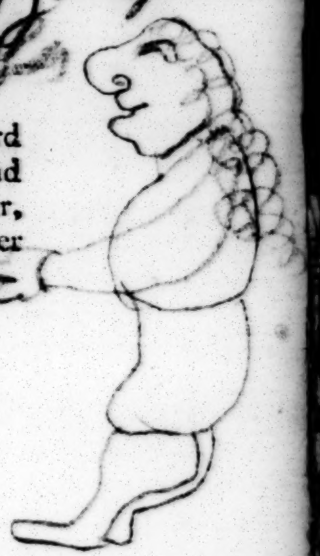
If the firſt of four magnitudes have the ſame ratio which the third has to the fourth; then any equimultiples whatever of the firſt and third ſhall have the ſame ratio to any equimultiples of the ſecond and fourth, viz. the equimultiple of the firſt ſhall have the ſame ratio to that of the ſecond which the equimultiple of the third has to that of the fourth.

Suppoſe $\mathrm{A} : \mathrm{B} = \mathrm{C} : \mathrm{D}$; then $m\,\mathrm{A} : n\,\mathrm{B} = m\,\mathrm{C} : n\,\mathrm{D}$.

$$\left.\begin{array}{r} m\,\mathrm{A} \\ m\,\mathrm{C} \end{array}\right] p \qquad \left.\begin{array}{r} n\,\mathrm{B} \\ n\,\mathrm{D} \end{array}\right] q \qquad \begin{array}{r|l} p\,m \times \mathrm{A} & q\,n \times \mathrm{B} \\ p\,m \times \mathrm{C} & q\,n \times \mathrm{D}. \end{array}$$

Since by Hypotheſis $\mathrm{A} : \mathrm{B} = \mathrm{C} : \mathrm{D}$ $\because$ Def. V.

If $p \times m\,\mathrm{A} > r =$ or $> s\ q \times n\,\mathrm{B}$; then $p \times m\,\mathrm{C} > r =$ or $> s\ q \times n\,\mathrm{B}$; $\because$ Converſe, Def. V. $m\,\mathrm{A} : n\,\mathrm{B} = m\,\mathrm{C} : n\,\mathrm{D}$.



COROLLARY.

Likewiſe, if the firſt has the ſame ratio to the ſecond which the third has to the fourth; then alſo any equimultiples whatever of the firſt and third have the ſame ratio to the ſecond and fourth; and, in like manner, the firſt and the third have the ſame ratio to any equimultiples whatever of the ſecond and fourth.

Suppoſe $\mathrm{A} : \mathrm{B} = \mathrm{C} : \mathrm{D}$; then $m\,\mathrm{A} : \mathrm{B} = m\,\mathrm{C} : \mathrm{D}$,
And $\mathrm{A} : n\,\mathrm{B} = \mathrm{C} : n\,\mathrm{D}$.

$\mathrm{A} : \mathrm{B} = \mathrm{C} : \mathrm{D}$; then $m\,\mathrm{A} : \mathrm{B} = m\,\mathrm{C} : \mathrm{D}$.

$$\left.\begin{array}{r} m\,\mathrm{A} \\ m\,\mathrm{C} \end{array}\right\} p \qquad \left.\begin{array}{r} \mathrm{B} \\ \mathrm{D} \end{array}\right\} q.\ \text{Def. V.} \left\{\begin{array}{l} \text{If } p\,m\,\mathrm{A} > r = \text{or} > s\ q\,\mathrm{B} \\ \text{Then } p\,m\,\mathrm{C} > r = \text{or} > s\ q\,\mathrm{D} \end{array}\right.$$

$\because$ Def. V. $m\,\mathrm{A} : \mathrm{B} = m\,\mathrm{C} : \mathrm{D}$.

$\mathrm{A} : \mathrm{B} = \mathrm{C} : \mathrm{D}$; then $\mathrm{A} : n\,\mathrm{B} = \mathrm{C} : n\,\mathrm{D}$.

$$\left.\begin{array}{r} \mathrm{A} \\ \mathrm{C} \end{array}\right\} p \qquad \left.\begin{array}{r} n\,\mathrm{B} \\ n\,\mathrm{D} \end{array}\right\} q.\ \text{Def. V.} \left\{\begin{array}{l} \text{If } p\,\mathrm{A} > r = \text{or} > s\ q\,n\,\mathrm{B} \\ \text{Then } p\,\mathrm{C} > r = \text{or} > s\ q\,n\,\mathrm{D} \end{array}\right.$$

$\because$ Def. V. $\mathrm{A} : n\,\mathrm{B} = \mathrm{C} : n\,\mathrm{D}$.

PROPOSITION V.—THEOREM.

If one magnitude be the ſame multiple of another, which a magnitude taken from the firſt is of a magnitude taken from the other, the remainder ſhall be the ſame multiple of the remainder that the whole is of the whole.

$$\left.\begin{array}{r|c|l} m\,\mathrm{A} + m\,\mathrm{B} & \mathrm{A} + \mathrm{B} & m \times \overline{\mathrm{A} + \mathrm{B}} \\ m\,\mathrm{A} & \mathrm{A} & m \times \mathrm{A} \\ \hline m\,\mathrm{B} & \mathrm{B} & m \times \mathrm{B} \end{array}\right\} \text{or} \left\{\begin{array}{r|c|l} m\,\mathrm{A} & \mathrm{A} & m \times \mathrm{A} \\ m\,\mathrm{B} & \mathrm{B} & m \times \mathrm{B} \\ \hline m\,\mathrm{A} - m\,\mathrm{B} & \mathrm{A} - \mathrm{B} & m \times \overline{\mathrm{A} - \mathrm{B}} \end{array}\right.$$

PROPOSITION VI.—THEOREM.

If two magnitudes be equimultiples of two others, and if equimultiples of theſe be taken from the firſt two, the remainders are either equal to theſe others, or equimultiples of them.

$$\begin{array}{c|c|c|l|l} m\,A & A & n\,A & m\,A - n\,A = \overline{m-n} \times A & \text{If } m-n = 1\text{, then } \overline{m-n} \times A = A \\ m\,B & B & n\,B & m\,B - n\,B = \overline{m-n} \times B & \text{And } \overline{m-n} \times B = B. \end{array}$$

When $m - n = 1$, if m be even then n muſt be odd.

PROPOSITION A.—THEOREM.

If the firſt of four magnitudes has to the ſecond the ſame ratio which the third has to the fourth; then if the firſt be greater than the ſecond, the third is alſo greater than the fourth; and if equal, *equal*; if leſs, *leſs*.

$$A : B = C : D$$

$$\left.\begin{matrix} A \\ C \end{matrix}\right\} m \quad \left.\begin{matrix} B \\ D \end{matrix}\right\} m \qquad \begin{array}{c|c} m\,A & m\,B \\ m\,C & m\,D \end{array} \quad \because \text{Def. V.}$$

If $m\,A > r = \text{or} > s\ m\,B$, then $m\,C > r = \text{or} > s\ m\,D \because$ (dividing by m), if $A > r = \text{or} > s\ B$, then $C > r = \text{or} > s\ D$.

PROPOSITION B.—THEOREM.

If four magnitudes are proportionals, they are proportionals alſo when taken inverſely.

Suppoſe $A : B = C : D$; then by inverſion, $B : A = D : C$.

$$\left.\begin{matrix} A \\ C \end{matrix}\right\} m \quad \left.\begin{matrix} B \\ D \end{matrix}\right\} n \ \text{Def. V.} \ \begin{matrix} m\,A > r = \text{or} > s\ n\,B \\ m\,C > r = \text{or} > s\ n\,D. \end{matrix}$$

$$\text{Or} \left\{\begin{matrix} n\,B > r = \text{or} > s\ m\,A \\ n\,D > r = \text{or} > s\ m\,C \end{matrix}\right. \quad \because \text{Def. V. } B : A = D : C.$$

PROPOSITION C.—THEOREM.

If the firſt be the ſame multiple of the ſecond, or the ſame part of it that the third is of the fourth, the firſt is to the ſecond as the third is to the fourth.

Caſe I. $\left.\begin{matrix} A = p\,B \\ C = p\,D \end{matrix}\right\}$ Then $A : B = C : D$.

$$\left.\begin{matrix} A \\ C \end{matrix}\right\} m \quad \left.\begin{matrix} B \\ D \end{matrix}\right\} n \quad \left.\begin{matrix} m\,A \\ m\,C \end{matrix}\right\} = \left\{\begin{matrix} m\,p\,B \\ m\,p\,D \end{matrix}\right. \quad \begin{array}{c|c} m\,p\,B & n\,B \\ m\,p\,D & n\,D. \end{array}$$

If $m > r = \text{or} > s\,n$; then $m\,B > r = \text{or} > s\,n\,B$, and $m\,D > r = \text{or} > s\,n\,D$.

As

$$\left.\begin{array}{r}\text{As alſo } mp\,B > r = \text{or} > s\,n\,B\\ \text{And } mp\,D > r = \text{or} > s\,n\,D\end{array}\right\}\text{ or }\begin{array}{l}m\,A > r = \text{or} > s\,n\,B\\ m\,C > r = \text{or} > s\,n\,D\end{array}\ \because\ \text{Def. V.}$$

$$A : B = C : D.$$

$$\text{Caſe II.}\quad\left.\begin{array}{r}A = \frac{1}{p}B\\ C = \frac{1}{p}D\end{array}\right\}\text{Then } A : B = C : D.$$

$$\left.\begin{array}{r}\frac{1}{p}B = A\\ \frac{1}{p}D = C\end{array}\right\}\text{ Hyp. or }\left\}\begin{array}{l}B = p\,A\\ D = p\,C\end{array}\right.\left\{\begin{array}{l}\because\ \text{Caſe I. } B : A = D : C\text{; and}\\ \text{inverſely, } A : B = C : D\ \text{(Prop. B.)}\end{array}\right.$$

PROPOSITION D.—THEOREM.

If the firſt be to the ſecond as the third to the fourth, and if the firſt be a multiple or part of the ſecond, the third is the ſame multiple, or the ſame part of the fourth.

Caſe I. Suppoſe $A : B = C : D$, and that $A = m\,B$; then $C = m\,D$. For, take $E = A\ (= m\,B)$ and $F = m\,D$. Since $A : B = C : D$, and $E = m\,B$; as alſo $F = m\,D$ (Hyp.) $\because A : E = C : F$ (Prop. IV. Corol.), but $A = E$ Hyp. $\because C = F$ (Prop. A); that is, $A = m\,B$, and $\because C = m\,D$.

Caſe II. Suppoſe $A : B = C\ D$, and that $A = \frac{1}{m}B$; then $C = \frac{1}{m}D$. Since $A : B = C : D$ (Hyp.) $\because$ Inverſely, $B : A = D : C$ (Prop. B); but $A = \frac{1}{m}B$ (Hyp.) or $m\,A = B$, and $\because$ (Caſe I.) $D = m\,C$ or $C = \frac{1}{m}D$; that is, $A = \frac{1}{m}B$, and $\because C = \frac{1}{m}D$.

COROLLARY.

In any analogy the firſt term has as much magnitude when compared with the ſecond, as the third when compared with the fourth; and, therefore, if the firſt term be greater, equal, or leſs than the ſecond, the third term will be equally greater, equal, or greatly leſs than the fourth.

PROPOSITION VII.—THEOREM.

Equal magnitudes have the ſame ratio to the ſame magnitude, and the ſame has the ſame ratio to equal magnitudes.

Caſe I. Suppoſe $A = B$, and that C is any other magnitude of the ſame kind; then $A : C = B : C$.

$$\left.\begin{array}{c}A\\ B\end{array}\right\}m\qquad\left.\begin{array}{c}C\\ C\end{array}\right\}n\qquad\left.\begin{array}{r}A = B\\ m\,A = m\,B\end{array}\right\}\text{Hyp.}\ \because$$

$$\left.\begin{array}{l}\text{If } m\,A > r = \text{or} > s\,n\,C\\ \text{Then } m\,B > r = \text{or} > s\,n\,C\end{array}\right\}\text{And}\ \because\ \text{Def. V. } A : C = B : C,$$

Caſe II. $C : A = C : B$

$$\left.\begin{matrix}C\\C\end{matrix}\right\}n\quad\left.\begin{matrix}A\\B\end{matrix}\right\}m\quad\left.\begin{matrix}A=B\\mA=mB\end{matrix}\right\}\begin{matrix}\text{Hyp.} \because \text{If } nC > r = \text{or} > s\ mA\\ \text{Then } nC > r = \text{or} > s\ mB\,;\end{matrix}$$

And conſequently, Def. V. $C : A = C : B$.

PROPOSITION VIII.—THEOREM.

Of unequal magnitudes the greater has a greater ratio to the ſame than the leſs has, and the ſame magnitude has a greater ratio to the leſs than it has to the greater.

$$\text{Hyp.}\left\}\begin{matrix}A > r\ B\\ C \text{ any third magnitude}\\ \text{of the ſame kind :}\end{matrix}\right\}\text{ Then }\left\{\begin{matrix}A : C > r\ B : C\\ C : B > r\ C : A.\end{matrix}\right.$$

Since $A > r\ B$ (Hyp.), the difference of A, B is $A - B$. Then, if the magnitude *which* is *not* the greater of the two $A - B$, B be not *leſs* than C, take $2\,\overline{A-B}$, $2\,B$, the doubles of $A-B$, B; but if that which is *not* the *greater* of the two $A-B$, B be leſs than C, it may be multiplied ſo as to become greater than C, whether it be $A-B$ or B. Let it be multiplied, therefore, until it become *greater* than C; and let the other be multiplied *as often*. Let $m\,\overline{A-B}$, $m\,B$, be theſe equimultiples, which are, therefore, each *greater* than C: And in every one of the caſes, let $q\,C$ be that multiple of C which *firſt* becomes greater than $m\,B$ and $p\,C$ $(=\overline{q-1}\times C)$, the multiple of C which is *next leſs* than $q\,C$.

Becauſe $q\,C$ is the multiple of C, which is the *firſt* that becomes $> r\ m\,B$; the next preceding multiple $p\,C$ is *not* $> r\ m\,B$; that is, $m\,B$ is *not* $> s\ p\,C$. And $m.\overline{A-B}$, $m\,B$, being equimultiples of the magnitudes $A-B$, B; the ſum $(m\,\overline{A-B}+m\,B=)\ m\,A$ and $m\,B$, are the ſame equimultiples of A B (Prop. I.), as is evident alſo by the notation.

But it was ſhown that $m\,B$ *not* $> s\ p\,C$, and $m\,\overline{A-B} > r\ C$ $\because$ the whole $(m\,B+m\,\overline{A-B}=)\ m\,A > r\ (p\,C+C=\overline{p+1}\times C=)\ q\,C$ (Hyp.); but as ſhown, $q\,C > r\ m\,B$, or $m\,B$ not $> r\ q\,C$. Thus:

$$\begin{matrix}A,\ C,\ B,\ C\\ \left.\begin{matrix}A\\B\end{matrix}\right\}m\quad\left.\begin{matrix}C\\C\end{matrix}\right\}q\end{matrix}\quad\left\{\begin{matrix}m\,A > r\ q\,C\\ m\,B \text{ not} > r\ q\,C\end{matrix}\right\}\left\{\begin{matrix}\because \text{Def. VII.}\\ A : C > r\ B : C.\end{matrix}\right.$$

Alſo, $C : B > r\ C : A$. For having ſtated the ſame multiples, &c. as above, it may be ſhown, in like manner, that $q\,C > r\ m\,B$, and *not* $> r\ m\,A$.

$$\begin{matrix}C,\ B,\ C,\ A\\ \left.\begin{matrix}C\\C\end{matrix}\right\}q\quad\left.\begin{matrix}B\\A\end{matrix}\right\}m\end{matrix}\quad\left.\begin{matrix}q\,C > r\ m\,B\\ q\,C \text{ not} > r\ m\,A\end{matrix}\right\}\left\{\begin{matrix}\because \text{Def. VII.}\\ C : B > r\ C : A.\end{matrix}\right.$$

Otherwiſe *.

$$\left.\begin{matrix}\overline{A+B} > r\ A\\ C\end{matrix}\right\}\text{ Then }\left\{\begin{matrix}A+B : C > r\ A : C\\ C : A > r\ C : \overline{A+B}.\end{matrix}\right.$$

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* *Mr Playfair's Geometry, Book V. p.* 142.

Take m, n, multipliers, ſuch that $m\,A > r\,C$, $m\,B > r\,C$, and $n\,C$ the leaſt multiple of C, exceeding $m\,A+m\,B$ $(=m\times\overline{A+B})$.

$$\left.\begin{array}{l}\text{Hyp. } n\,C > r\ m\,A+m\,B\\ \therefore \overline{n-1}\ C > s\ m\,A+m\,B\end{array}\right\}\ \text{or}\ \left\{m\,A+m\,B > r\ \overline{n-1}.C.\right.$$

$$\left.\begin{array}{l}\text{Hyp. } n\,C > r\ m\,A+m\,B\\ \because (n\,C-C=)\,n-1.C > s\ m\,A+\overline{m\,B-C}\end{array}\right]\quad\left[\begin{array}{l}\text{Hyp. } m\,B > r\ C\\ \because \overline{n-1}\ C > r\ m\,A.\end{array}\right.$$

Thus:

$$(m\,A+m\,B=)\ m.\overline{A+B} > r\ \overline{n-1}.C$$

$$m\,A\ (> s\ \text{and})\ \text{not} > r\ \overline{n-1}\ C$$

$$\because \text{Def. VII. } \overline{A+B} : C > r\ A : C.$$

Alſo, as $\overline{n-1}\ C > r\ m\,A$

$$\overline{n-1}\ C\ (> s\ \text{and})\ \text{not} > r\ m\,\overline{A+B}$$

$$\because \text{Def. VII. } C : A > r\ C : \overline{A+B}.$$

PROPOSITION IX.

Magnitudes which have the ſame ratio to the ſame magnitude, are equal to one another; and thoſe to which the ſame magnitude has the ſame ratio, are equal to one another.

$$\text{Hyp.}\ \left[\begin{array}{l}A : C = B : C\\ C : B = C : A\end{array}\right]\ \text{Then } A = B.$$

If A, B are not equal, they muſt be *unequal*, and one *neceſſarily* greater than the other, which ſuppoſe to be A. Then ſince $A > r\ B$, and C a third magnitude of the ſame kind: there are (Prop. VIII.) *ſome* equimultiples $m\,A$, $m\,B$ of A, B, and ſome equimultiple $q\,C$ of C, ſuch that $m\,A > r\ q\,C$ and $m\,B$ *not* $> r\ q\,C$, or $q\,C > r\ m\,B$ and *not* $> r\ m\,A$. But, becauſe $A : C = B : C$ and $C : B = C : A$ (Hyp.) $\because$ Def. V. if $m\,A > r\ q\,C$, then $m\,B > r\ q\,C$; or if $q\,C > r\ m\,B$, then $q\,C > r\ m\,A$; but $m\,B$ *not* $> r\ q\,C$, and $q\,C$ *not* $> r\ m\,A$, which are impoſſible. Therefore, ſince the ſuppoſed inequality of A, B involves an *impoſſibility* in both caſes, the magnitudes of A, B are not unequal, that is, they are equal.

PROPOSITION X.

That magnitude which has a greater ratio than another has unto the ſame magnitude, is the greater of the two; and that magnitude to which the ſame has a greater ratio than it has unto another magnitude, is the leſs of the two.

$$\text{Hyp.}\ \left\{\begin{array}{l}A : C > r\ B : C\\ C : B > r\ C : A\end{array}\right.\ \left\{\text{Then } A > r\ B.\right.$$

Becauſe $A : C > r\ B : C$, and $C : B > r\ C : A$ (Hyp.): there are *ſome* (Def. VII.) equimultiples $m\,A$, $m\,B$ of A B, and *ſome* multiple $q\,C$ of C, ſuch that $m\,A > r\ q\,C$ and $m\,B$ *not* $> r\ q\,C$, or $q\,C > r\ m\,B$ and *not* $> r\ m\,A$. Therefore, ſince $m\,A > r\ m\,B$ in both caſes, it is $A > r\ B$ (Axiom IV.).

PROPOSITION XI.

Ratios that are the ſame to the ſame ratio, are the ſame to one another.

$$\text{Hyp.}\ \left[\begin{array}{l} A : B = C : D \\ C : D = E : F \end{array}\right]\ \text{Then } A : B = E : F.$$

$$\left.\begin{array}{l}A\\C\\E\end{array}\right\}m \quad \left.\begin{array}{l}B\\D\\F\end{array}\right\}n \quad \left.\begin{array}{c} A : B = C : D\ \text{Hyp.} \\ \because \text{Def. V.} \\ mA > r = \text{or} > s\ nB \\ mC > r = \text{or} > s\ nD\end{array}\right\}\left\{\begin{array}{c} C : D = E : F\ \text{Hyp.}\\ \because \text{Def. V.}\\ mC > r = \text{or} > s\ nD \\ mE > r = \text{or} > s\ nF.\end{array}\right\}$$

$$\begin{array}{l}\text{Thus: when } mC > r = \text{or} > s\ nD \\ \quad \left.\begin{array}{l} mA > r = > s\ nB \\ mE > r = > s\ nF\end{array}\right\} \begin{array}{l}\because \text{Def. V.}\\ A : B = E : F.\end{array}\end{array}$$

COROLLARY.

Ratios that are the ſame to the ſame ratios, are the ſame to one another.

PROPOSITION XII.—THEOREM.

If any number of magnitudes be proportionals, as one of the antecedents is to its conſequent, ſo ſhall all the antecedents taken together be to all the conſequents.

$$\text{Hyp.}\ \big|\ A : B = C : D = E : F, \text{\&c.}\ \big|\ \text{Then } A : B = A + C + E, \text{\&c.} : B + D + F, \text{\&c.}$$

$$A : B = C : D = E : F, \text{\&c. (Hyp.)}$$

$$\left.\begin{array}{c}A\\C\\E\\\text{\&c.}\end{array}\right\}m \qquad \left.\begin{array}{c}B\\D\\F\\\text{\&c.}\end{array}\right\}n$$

And by Def. V.

$$\begin{array}{l}\text{If } mA > r = \text{or} > s\ nB \\ \hline \text{Then } mC > r = \text{or} > s\ nD \\ \quad\ \ mE > r = \text{or} > s\ nF \\ \quad\ \ \text{\&c.} \qquad\qquad\qquad \text{\&c.} \\ \hline \end{array}$$

$$\text{And conſequently, } mA + mC + mE \text{ \&c.} > r = \text{or} > s\ nB + nD + nF \text{ \&c.}$$

$$\text{Or } m \times \overline{A + C + E \text{ \&c.}} > r = \text{or} > sn \times \overline{B + D + F \text{ \&c.}}$$

Therefore, by Def. V.

$$A : B\ (= C : D = E : F \text{ \&c.}) = A + C + E \text{ \&c.} : B + D + F \text{ \&c.}$$

PROPOSITION

PROPOSITION XIII.—THEOREM.

If the firſt has to the ſecond the ſame ratio which the third has to the fourth, but the third has to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth has to the ſixth.

$$\text{Hyp.}\left\{\begin{array}{l} A : B = C : D \\ C : D > r\, E : F \end{array}\right\} \text{Then } A : B > r\, E : F.$$

Since $C : D > r\, E : F$ (Hyp.), there are (Def. VII.) *ſome* equimultiples $m\,C$, $m\,E$, of C, E, and *ſome* equimultiples of $n\,D$, $n\,F$, of D, F, ſuch that $m\,C > r\, n\,D$, and $m\,E$ *not* $> r\, n\,F$.

$$\left.\begin{array}{l} \text{Alſo, ſince } A : B = C : D, \text{ Hyp.} \\ \quad \left.\begin{array}{l} A \\ C \end{array}\right\} m \quad \left.\begin{array}{l} B \\ D \end{array}\right\} n \end{array}\right\} \begin{array}{l} \quad \because \text{Def. V.} \\ \text{If } m\,A > r = \text{or} > s\; n\,B \\ \text{Then } m\,C > r = \text{or} > s\; n\,D. \end{array}$$

$$\left.\begin{array}{r} \text{By ſuppoſition, } m\,C > r\; n\,D \\ \because \text{Def. V.} \quad m\,A > r\; n\,B \end{array}\right\} \left.\begin{array}{r} \text{Thus } m\,A > r\; n\,B \\ m\,E \text{ not} > r\; n\,F \end{array}\right\} \begin{array}{l} \because \text{Def. VII.} \\ A : B > r\, E : F. \end{array}$$

COROLLARY I.

If the firſt has a greater ratio to the ſecond than the third has to the fourth, but the third has the ſame ratio to the fourth which the fifth has to the ſixth; it may be demonſtrated, in like manner, that the firſt has a greater ratio to the ſecond than the fifth has to the ſixth.

$$\text{Hyp.}\left\{\begin{array}{l} A : B > r\, C : D \\ C : D = E : F \end{array}\right\} \text{Then } A : B > r : E : F.$$

COROLLARY II.

If the firſt has the ſame ratio to the ſecond which the third has to the fourth, but the third has a leſs ratio to the fourth than the fifth has to the ſixth; then the firſt has a leſs ratio to the ſecond than the fifth has to the ſixth.

$$\text{Hyp.}\left\{\begin{array}{l} A : B = C : D \\ C : D > s\, E : F \end{array}\right\} \text{Then } A : B > s\, E : F.$$

COROLLARY III.

If the firſt has a greater or leſs ratio to the ſecond than the third has to the fourth, and the third has a greater or leſs ratio to the fourth than the fifth has to the ſixth; then the firſt has a greater or leſs ratio to the ſecond than the fifth has to the ſixth.

$$\text{Hyp.}\left\{\begin{array}{l} A : B > r \text{ or } > s\; C : D \\ C : D > r \text{ or } > s\; E : F \end{array}\right\} \text{Then } A : B > r \text{ or } > s\; E : F.$$

PROPOSITION XIV.—THEOREM.

If the firſt has to the ſecond the ſame ratio which the third has to the fourth; then, if the firſt be greater than the third, the ſecond ſhall be greater than the fourth; and if equal, equal; and if leſs, leſs.

$$\text{Hyp.}\ \left\{A : B = C : D\right\} \begin{cases}\text{If } A \rhd r = \text{or} \rhd s\ C \\ \text{Then } B \rhd r = \text{or} \rhd s\ D.\end{cases}$$

I. $A \rhd r\ C$, Hyp.
$\because A : B \rhd r\ C : B$, Prop. VIII.
$A : B = C : D$, Hyp.
$\because C : D \rhd r\ C : B$, Prop. XIII.
$\because D \rhd s\ B$, Prop. X.
Or $B \rhd r\ D$.

II. $A = C$, Hyp.
$A : B = C : D$, Hyp.
Or $A : B = A : D$.
$\because B = D$, Prop. IX.

III. $A \rhd s\ C$, Hyp.
$\because C \rhd r\ A$
$C : D = A : B$, Hyp.
$\because D \rhd r\ B$, Caſe I.
Or $B \rhd s\ D$.

PROPOSITION XV.—THEOREM.

Magnitudes have the ſame ratio to one another which their equimultiples have.

$A : B = mA : mB$, m being *any* number.
A, A, A, A, &c. to m terms $= mA$.
B, B, B, B, &c. to m terms $= mB$.
$A = A$, $B = B$.
$A : B = A : B = A : B$, &c. to m terms.
$\because A : B = mA : mB$ (Prop. XII.).

COROLLARY.

Magnitudes have the ſame ratio to one another which their equiſubmultiples have.

$$A : B = \frac{1}{m}A : \frac{1}{m}B.$$

$$\text{Suppoſe}\left\{\begin{matrix} C = \frac{1}{m}A \\ D = \frac{1}{m}B \end{matrix}\right\}\ \text{Then}\ \left\{\begin{matrix} mC = A \\ mD = B \end{matrix}\right\}$$

$$mC : mD = C : D\ (\text{Prop.})$$

$$\text{Or } A : B = \frac{1}{m}A : \frac{1}{m}B.$$

SCHOLIUM.

The Corollary, though not given by Euclid, is equally uſeful with the Propoſition.

PROPOSITION XVI.—THEOREM.

If four magnitudes of the ſame kind be proportionals, they ſhall alſo be proportionals when taken alternately.

Hyp. | $A : B = C : D$ | *Alternately*, $A : C = B : D$.

$\left.\begin{matrix}A\\B\end{matrix}\right\}m$

$A : B = mA : mB$, Prop. XV.

$A : B = C : D$, Hyp.

$\because C : D = mA : mB$, Prop. XI.

$\left.\begin{matrix}C\\D\end{matrix}\right\}n$

$C : D = nC : nD$, Prop. XV.

$\because mA : mB = nC : nD$, Prop. XI.

$\left.\begin{matrix}\because \text{If } mA > r = \text{or} > snC\\ \text{Then } mB > r = \text{or} > snD\end{matrix}\right\}$ Pr. XIV.

And $\because$ Def. V. $A : C = B : D$.

PROPOSITION XVII.—THEOREM.

If magnitudes taken jointly be proportionals, they ſhall alſo be proportionals when taken disjointly; that is, if two magnitudes together have to one of them the ſame ratio which two others have to one of theſe, the remaining one of the firſt two ſhall have to the other the ſame ratio which the remaining one of the laſt two has to the other of theſe.

Hyp. | $A+B : B = C+D : D$ | *Disjointly*, $A : B = C : D$.

$\left.\begin{matrix}A\\B\\C\\D\end{matrix}\right\}m$ $\left.\begin{matrix}B\\D\end{matrix}\right\}n$

$$\left.\begin{matrix}mA = m\times A\\ mB = m\times B\\ mA+mB = m\times\overline{A+B}\\ \text{But } \overline{A+B} > rB\\ \because m\times\overline{A+B} > rmB\end{matrix}\right\}\left\{\begin{matrix}mC = m\times C\\ mD = m\times D\\ mC+mD = m\times\overline{C+D}\\ \text{But } \overline{C+D} > rD\\ \because m\times\overline{C+D} > rmD\end{matrix}\right.$$

$A+B : B = C+D : D$, Hyp.

$\because$ Def. V.

$\left.\begin{matrix}\text{If } m.\overline{A+B} > r = \text{or} > snB\\ \text{Then } m.\overline{C+D} > r = \text{or} > snD\end{matrix}\right\}$

$\left\{\begin{matrix}\text{Whence, taking } mB \text{ from the former, and } mD \text{ from the latter,}\\ \text{If } mA > r = \text{or} > s\,\overline{n-m}\times B\\ \text{Then } mC > r = \text{or} > s\,\overline{n-m}\times D.\end{matrix}\right.$

And $\because$ Def. V. $A : B = C : D$.

PROPOSITION XVIII.—THEOREM.

If magnitudes taken disjointly be proportionals, they ſhall alſo be proportionals when taken jointly; that is, if the firſt be to the ſecond as the third to the fourth, the firſt and ſecond together ſhall be to the ſecond as the third and fourth together to the fourth.

Hyp. | $A-B : B = C-D : D$ | Jointly, $A : B = C : D$.

$\left.\begin{matrix}A\\B\\C\\D\end{matrix}\right\}m$ $\left.\begin{matrix}B\\D\end{matrix}\right\}n$

$$\left.\begin{array}{r} mA = m\times A \\ mB = m\times B \\ mA - mB = m\times\overline{A-B}\end{array}\right\}\quad\left\{\begin{array}{r} mC = m\times C\\ mD = m\times D\\ mC - mD = m\times\overline{C-D}\end{array}\right.$$

$$A-B : B = C-D : D, \text{ Hyp.}$$

$$\left.\begin{array}{r}\because \text{Def. V.}\\ \text{If } m.\overline{A-B} > r = \text{or} > s\ nB\\ \text{Then } m.\overline{C-D} > r = \text{or} > s\ nD\end{array}\right\}\begin{array}{l}\text{Whence, adding } mB \text{ to the former, and}\\ \quad mD \text{ to the latter,}\\ \text{If } mA > r = \text{or} > s\ \overline{m+n}\times B\\ \text{Then } mC > r = \text{or} > s\ \overline{m+n}\times D\end{array}$$

And $\because$ Def. V. $A : B = C : D$.

PROPOSITION XIX.—THEOREM.

If a whole magnitude be to a whole as a magnitude taken from the firſt is to a magnitude taken from the other, the remainder ſhall be to the remainder as the whole to the whole.

Hyp. $A+B : C+D = B : D$ | Then $A : C = A+B : C+D$.
Since $A+B : C+D = B : D$, Hyp.
$\because$ Alternately, $A+B : B = C+D : D$, Prop. XVI.
Disjointly, $A : B = C : D$, Prop. XVII.
Alternately, $A : C = B : D$, Prop. XVI.
Hyp. $A+B : C+D = B : D$.
$\because$ $A : C = A+B : C+D$, Prop. XI.

COROLLARY.

If the whole be to the whole as a magnitude taken from the firſt is to a magnitude taken from the other, the remainder likewiſe is to the remainder as the magnitude taken from the firſt to that taken from the other.

Hyp. | $A+B : C+D = B : D$ | Then $A : C = B : D$.
The fourth ſtep of the demonſtration.

PROPOSITION E.—THEOREM.

If four magnitudes be proportionals, they are alſo proportionals by converſion; that is, the firſt is to its exceſs above the ſecond as the third to its exceſs above the fourth.

Hyp. | $A+B : B = C+D : D$ | Converſely, $A+B : A = C+D : C$.
Since $A+B : B = C+D : D$, Hyp.
$\because$ Disjointly, $A : B = C : D$, Prop. XVII.
Inverſely, $B : A = D : C$, Prop. B.
Jointly, $A+B : A = C+D : C$, Prop. XVIII.

PROPOSITION

PROPOSITION XX.—THEOREM.

If there be three magnitudes, and other three, which taken two and two ordinately (Def. **XVIII, XIX.**), have the ſame ratio; if the firſt be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

$$\text{Hyp.}\left\{\begin{matrix}A, B, C\\ D, E, F\end{matrix}\right\}\left.\begin{matrix}A : B = D : E\\ B : C = E : F\end{matrix}\right\}\begin{matrix}\text{If } A \vartriangleright r = \text{or} \vartriangleright s\ C\\ \text{Then } D \vartriangleright r = \text{or} \vartriangleright s\ F.\end{matrix}$$

I. A ⊳ *r* C, Hyp.
∴ A : B ⊳ *r* C : B, Prop. VIII.
But D : E = A : B, Hyp.
∴ D : E ⊳ *r* C : B, Prop. XIII.
And B : C = E : F, Hyp.
Or C : B = F : E, Prop. B.
∴ D : E ⊳ *r* F : E, Prop. XIII. Cor.
And ∴ D ⊳ *r* F, Prop.

II. A = C, Hyp.
∴ A : B = C : B, Prop. VII.

$$\left.\begin{matrix}\text{But } A : B = D : E\\ C : B = F : E, \text{Prop. B.}\end{matrix}\right\}\text{Hyp.}$$

∴ D : E = F : E, Prop. XI.
∴ D = F, Prop. IX.

III. A ⊳ *s* C, Hyp.
∴ C ⊳ *r* A
C : B = F : E, Caſe I.
B : A = E : D, Hyp. Prop. B.

$$\left\{\begin{matrix}C, B, A\\ F, E, D\end{matrix}\right\}\left\{\begin{matrix}C : B = F : E\\ B : A = E : D\end{matrix}\right\}$$

∴ F ⊳ *r* D, Caſe I.
Or D ⊳ *s* F.

PROPOSITION XXI.—THEOREM.

If there be three magnitudes, and other three, which taken two and two *inordinately* (Def. XX.), have the ſame ratio; if the firſt magnitude be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

$$\text{Hyp.}\begin{pmatrix}A, B, C\\ D, E, F\end{pmatrix}\begin{pmatrix}A : B = E : F\\ B : C = D : E\end{pmatrix}\begin{matrix}\text{If } A \vartriangleright r = \text{or} \vartriangleright s\ C\\ \text{Then } D \vartriangleright r = \text{or} \vartriangleright s\ F.\end{matrix}$$

I. A ⊳ *r* C, Hyp.
∴ A : B ⊳ *r* C : B, Prop. VIII.
But E : F = A : B, Hyp.
∴ E : F ⊳ *r* C : B, Prop. XIII.
And B : C = D : E, Hyp.
Or C : B = E : D, Prop. B.
∴ E : F ⊳ *r* E : D, Prop. XIII. Cor.
And ∴ F ⊳ *s* D, Prop. X.
Or D ⊳ *r* F.

II. A = C, Hyp.
∴ A : B = C : B, Prop. VII.
But A : B = E : F, Hyp.
C : B = E : D, Hyp. Prop. B.
∴ E : F = E : D, Prop. XI.
And ∴ D = F, Prop. IX.

III.

$$\text{III. } A > s\; C$$

$$\because C > r\; A$$

$$\left.\begin{matrix} A, B, C \\ D, E, F \end{matrix}\right) \begin{matrix} C : B = E : D, \text{ Hyp. Prop. B.} \\ B : A = F : E, \text{ Hyp. Prop. B.} \end{matrix}$$

$$\because F > r\; D, \text{ Case I.}$$

$$\text{Or } D > s\; F.$$

PROPOSITION XXII.—THEOREM.

If there be any number of magnitudes, and as many others, which taken two and two *ordinately*, have the ſame ratio; then, by equality (Def. XVIII, XIX.), the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt.

$$\text{Hyp. I.} \left\{\begin{matrix} A,B,C \\ D,E,F \end{matrix}\right\} \begin{matrix} A:B = D:E \\ B:C = E:F \end{matrix} \left\{\begin{matrix} \text{Equality} \\ A:C = D:F \end{matrix}\right\} \left\{\begin{matrix} A \\ D \end{matrix}\right\} m \left.\begin{matrix} B \\ C \end{matrix}\right\} n \left.\begin{matrix} C \\ F \end{matrix}\right\} p.$$

$$\text{Hyp.} \left\{\begin{matrix} A : B = D : E \\ B : C = E : F \end{matrix}\right\} \because \text{Prop. IV.} \left\{\begin{matrix} mA : nB = mD : nE \\ nB : pC = nE : pF \end{matrix}\right\}$$

$$\text{Thus} \left\{\begin{matrix} mA,\ nB,\ pC \\ mD,\ nE,\ pF \end{matrix}\right\} \text{And} \left\{\begin{matrix} mA : nB = mD : nE \\ nB : pC = nE : pF. \end{matrix}\right.$$

$$\because \text{Prop. XX.} \left\{\begin{matrix} \text{If } mA > r = \text{or} > s\; pC \\ \text{Then } mD > r = \text{or} > s\; pF \end{matrix}\right\} \because \text{Def. V.} \left\{ A : C = D : F, \right.$$

$$\text{Hyp. II.} \left\{\begin{matrix} A, B, C, D, \&c. \\ E, F, G, H, \&c. \end{matrix}\right. \left\{\begin{matrix} A : B = E : F \\ B : C = F : G \\ C : D = G : H \\ \&c.\ \&c. \end{matrix}\right\} \begin{matrix} \text{Equality.} \\ A : D = E : H. \end{matrix}$$

$$\left\{\begin{matrix} A,B,C \\ E,F,G \end{matrix}\right\} \left.\begin{matrix} A : B = E : F, \text{ Hyp.} \\ B : C = F : G, \text{ Hyp.} \end{matrix}\right\} \begin{matrix} \because \text{Case I.} \\ A : C = E : G \end{matrix} \left\{\begin{matrix} A : C = E : G \\ C : D = G : H, \text{ Hyp.} \end{matrix}\right.$$

$$\left.\begin{matrix} \because \text{Case I.} \\ A : D = E : H \end{matrix}\right\} \text{And ſo on, whatever be the number of magnitudes,}$$

PROPOSITION XXIII.—THEOREM.

If there be any number of magnitudes, and as many others, which taken two and two *inordinately*, have the ſame ratio; then, by perturberate equality (Def. XX.), the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt.

$$\text{Hyp. I.} \left(\begin{matrix} A, B, C \\ D, E, F \end{matrix}\right) \begin{matrix} A : B = E : F \\ B : C = D : E \end{matrix} \left(\begin{matrix} \text{Perturberate equality.} \\ A : C = D : F. \end{matrix}\right.$$

$$\left.\begin{matrix}A\\B\\D\end{matrix}\right\}m \qquad \left.\begin{matrix}C\\E\\F\end{matrix}\right\}n \qquad \text{Prop. XV.}\left(\begin{matrix}A:B=mA:mB\\E:F=nE:nF\end{matrix}\right)\begin{matrix}\because\ \text{Prop. XI.}\\mA:mB=nE:nF.\end{matrix}$$

$$\text{Hyp.}\left(B:C=D:E\right)\begin{matrix}\because\ \text{Prop. IV.}\\mB:nC=mD:nE.\end{matrix}$$

$$\left(\begin{matrix}mA,\ mB,\ nC\\mD,\ nE,\ nF\end{matrix}\right)\begin{matrix}mA:mB=nE:nF\\mB:nC=mD:nE\end{matrix}\left(\begin{matrix}\text{If } mA > r = \text{or} > s\ nC\\\text{Then } mD > r = \text{or} > s\ nF\end{matrix}\right)\begin{matrix}\text{Prop.}\\\text{XXI.}\end{matrix}$$

$$\because\ \text{Def. V.}\ A:C=D:F.$$

$$\text{Hyp. II.}\left(\begin{matrix}A,\ B,\ C,\ D,\ \&c.\\E,\ F,\ G,\ H,\ \&c.\end{matrix}\right)\left.\begin{matrix}A:B=G:H\\B:C=F:G\\C:D=E:F\\\&c.\ \&c.\end{matrix}\right\}\begin{matrix}\text{Perturberate equality.}\\A:D=E:H.\end{matrix}$$

$$\left(\begin{matrix}A,\ B,\ C\\F,\ G,\ H\end{matrix}\right)\begin{matrix}A:B=G:H,\ \text{Hyp.}\\B:C=F:G,\ \text{Hyp.}\end{matrix}\left(\begin{matrix}\because\ \text{Caſe I.}\\A:C=F:H\end{matrix}\right(\begin{matrix}A:C=F:H,\ \text{Caſe I.}\\C:D=E:F,\ \text{Hyp.}\end{matrix}$$

$$\left.\begin{matrix}\because\ \text{Caſe I.}\\A:D=E:H\end{matrix}\right)\text{And ſo on, whatever be the number of magnitudes.}$$

SCHOLIUM.

From this Propoſition and Propoſition XXI, as alſo from Propoſition V and Propoſition XX, it follows directly, that ratios compounded of the ſame or of equal ratios, are the ſame with one another; as particularly demonſtrated by Dr Simſon, in Propoſitions F, G, H, I, and K, following Propoſition XXV, which are nothing more than ſimple inferences or corollaries from this Propoſition and Propoſition XXI.

PROPOSITION XXIV.—THEOREM.

If the firſt has to the ſecond the ſame ratio which the third has to the fourth, and the fifth to the ſecond the ſame ratio which the ſixth has to the fourth; the firſt and fifth together ſhall have to the ſecond the ſame ratio which the third and ſixth together have to the fourth.

$$\text{Hyp.}\left(\begin{matrix}A:B=C:D\\E:B=F:D\end{matrix}\right)\begin{matrix}\text{Then}\\A+E:B=C+F:D.\end{matrix}$$

$$\begin{matrix}E:B=F:D,\ \text{Hyp.}\\\because\ B:E=D:F,\ \text{Prop. B.}\end{matrix}\left[\begin{matrix}A:B=C:D,\ \text{Hyp.}\\B:E=D:F,\ \text{Hyp. Prop. B.}\end{matrix}\right.$$

$$\left.\begin{matrix}\because\ \text{Equality, Prop. XXII.}\\A:E=C:F\end{matrix}\right]\begin{matrix}\because\ \text{Jointly, Prop. XVIII.}\\A+E:E=C+F:F.\end{matrix}$$

$$\left.\begin{matrix}\text{Thus } E:B=F:D,\ \text{Hyp.}\\\text{And } A+E\ ;\ E=C+F\ ;\ F\end{matrix}\right]\begin{matrix}\because\ \text{Equality, Prop. XXII.}\\A+E\ ;\ B=C+F:D.\end{matrix}$$

COROLLARY I.

If the ſame hypotheſis be made as in the Propoſition, the exceſs of the firſt and fifth ſhall be to the ſecond as the exceſs of the third and ſixth to the fourth.

That is, $A \backsim E : B = C \backsim F : D$.

The demonſtration of this is the ſame with that of the Propoſition, if terms be taken disjointly inſtead of jointly.

COROLLARY II.

The Propoſition holds true of two ranks of magnitudes, whatever be their numbers, of which each of the firſt rank has to a ſecond magnitude the ſame ratio that the correſponding one of the ſecond rank has to a fourth magnitude, as is manifeſt.

PROPOSITION XXV.—THEOREM.

If four magnitudes are proportionals, the greateſt and leaſt of them together are greater than the other two together.

$$\text{Hyp.}\left\{\begin{array}{l} G+H : K+L = H : L\,; \\ \text{where } G+H \text{ is the greateſt,} \\ \text{and } \because L \text{ the leaſt, Prop. A, XIV.}\end{array}\right\}\begin{array}{l}\text{Then} \\ \overline{G+H}+L \rhd r\,\overline{K+L}+H.\end{array}$$

Since $G+H : K+L = H : L$, Hyp.
$\because G : K = G+H : K+L$, Prop. XIX.

But $\overline{G+H} \rhd r\ \overline{K+L}$, Hyp.
$\because G \rhd r\ K$, Prop. A.

And $\because G+\overline{H+L} \rhd r\ K+\overline{H+L}$, Ax. IV. e. 1.

That is, $\overline{G+H+L} \rhd r\ \overline{K+L+H}$.

Otherwiſe.

$$\text{Hyp.}\left\{\begin{array}{l} A : B = C : D. \\ A \text{ greateſt, } D \text{ leaſt.}\end{array}\right\}\begin{array}{l}\text{Then} \\ \overline{A+D} \rhd r\ \overline{B+C}.\end{array}$$

Since $A : B = C : D$, Hyp.
$\because A-C : B-D = A : B$, Prop. XIX. Corol.
But $A \rhd r\ B$, Hyp.
$\because A-C \rhd r\ B-D$, Prop. A.
And $C+D = C+D$
$\because \overline{A+D} \rhd r\ \overline{B+C}$, Ax. IV. e. 1,

PROPOSITION F.—THEOREM.

Ratios which are compounded of the ſame ratios, are the ſame with one another.

$$\text{Hyp.}\left\{\begin{matrix}A,B,C\\D,E,F\end{matrix}\right.\left\{\begin{matrix}A:B=D:E\\B:C=E:F\end{matrix}\right\}\begin{matrix}\text{Then}\\A:C=D:F\end{matrix}\left\{\begin{matrix}A:B=E:F\\B:C=D:E\end{matrix}\right\}\begin{matrix}\text{Then}\\A:C=D:F.\end{matrix}$$

$$\times\ \text{Def.}$$

$$\text{Hyp.}\left\{\begin{matrix}A,B,C\\D,E,F\end{matrix}\right.\left\{\begin{matrix}A:B=D:E\\B:C=E:F\end{matrix}\right.\left\{\begin{matrix}\because\text{Prop. XXII.}\\A:C=D:F\end{matrix}\right\}\begin{matrix}A:C=\overline{A:B}+\overline{B:C}\\D:F=\overline{D:E}+\overline{E:F}.\end{matrix}$$

$$\times\ \text{Def.}$$

$$\text{Hyp.}\left\{\begin{matrix}A,B,C\\D,E,F\end{matrix}\right\}\begin{matrix}A:B=E:F\\B:C=D:E\end{matrix}\left\{\begin{matrix}\because\text{Prop. XXIII.}\\A:C=D:F\end{matrix}\right\}\left\{\begin{matrix}A:C=\overline{A:B}+\overline{B:C}\\D:F=\overline{E:F}+\overline{D:E}.\end{matrix}\right.$$

PROPOSITION G.—THEOREM.

If ſeveral ratios be the ſame with ſeveral ratios, each to each, the ratio which is compounded of ratios which are the ſame with the firſt ratios, each to each, is the ſame with the ratio compounded of ratios which are the ſame with the other ratios, each to each.

$$\text{Hyp.}\left\{\begin{matrix}A,B,C,D\\E,F,G,H\end{matrix}\right\}\begin{matrix}K,L,M\\N,O,P\end{matrix}\left.\right\}\begin{matrix}A:B=E:F,C:D=G:H\\A:B=K:L,C:D=L:M\end{matrix}\left\{\begin{matrix}E:F=N:O\\G:H=O:P\end{matrix}\right.$$

$$\text{Then}\ |\ K:M=N:P.\ |$$

$$\begin{matrix}K,L,M\\N,O,P\end{matrix}\left\{\begin{matrix}K:L(=A:B=E:F)=N:O,\ \text{Hyp.}\\L:M(=C:D=G:H)=O:P,\ \text{Hyp.}\end{matrix}\right\}\begin{matrix}\because\text{Prop. XXII.}\\K:M=N:P.\end{matrix}$$

$$\left\{\begin{matrix}K:M=\overline{K:L}+\overline{L:M},\text{Def. X.}=\overline{A:B}+\overline{C:D}\\N:P=\overline{N:O}+\overline{O:P},\text{Def. X.}=\overline{E:F}+\overline{G:H}\end{matrix}\right\}\text{Hyp.}$$

PROPOSITION H.—THEOREM.

If a ratio compounded of ſeveral ratios be the ſame with a ratio compounded of any other ratios, and if one of the firſt ratios, or a ratio compounded of any of the firſt, be the ſame with one of the laſt ratios, or with the ratio compounded of any of the laſt, then the ratio compounded of the remaining ratios of the firſt, or the remaining ratio of the firſt, if but one remain, is the ſame with the ratio compounded of thoſe remaining of the laſt, or with the remaining ratio of the laſt.

$$\text{Hyp.}\left\{\begin{matrix}A,B,C,D,E,F\\G,H,K,L,M\end{matrix}\right\}\begin{matrix}A:F=G:M\\A\underset{D}{:}D=G:K\end{matrix}\left.\right\}\begin{matrix}\text{Then}\\D:F=K:M.\end{matrix}$$

$$\left\{\begin{array}{l} \left.\begin{array}{l} A:F=\overline{A:B}+\overline{B:C}+\overline{C:D}+\overline{D:E}+\overline{E:F} \\ G:M=\overline{G:H}+\overline{H:K}+\overline{K:L}+\overline{L:M}\end{array}\right\} A:F=G:M,\text{ Hyp.} \\ \left.\begin{array}{l} A:D=\overline{A:B}+\overline{B:C}+\overline{C:D} \\ G:K=\overline{G:K}+\overline{H:K}\end{array}\right] A:D=G:K,\text{Hyp.}\ |\ D:A=K:G,\text{Pr.B.} \\ \hline \left.\begin{array}{l} D:F=\overline{D:E}+\overline{E:F}\\ K:M=\overline{K:L}+\overline{L:M}\end{array}\right\} \qquad \left\{\begin{array}{c}\because \text{Prop. XXII.}\\ D:F=K:M\end{array}\right\} \\ \hline \end{array}\right.$$

PROPOSITION K.—THEOREM.

If there be any number of ratios, and any number of other ratios, ſuch that the ratio compounded of ratios which are the ſame with the firſt ratios, each to each, is the ſame with the ratio compounded of ratios which are the ſame, each to each, with the laſt ratios; and if one of the firſt ratios, or the ratio which is compounded of ratios which are the ſame with ſeveral of the firſt ratios, each to each, be the ſame with one of the laſt ratios, or with the ratio compounded of ratios which are the ſame, each to each, with ſeveral of the laſt ratios; then the ratio compounded of ratios which are the ſame with the remaining ratios of the firſt, each to each, or the remaining ratio of the firſt, if but one remain, is the ſame with the ratio compounded of ratios which are the ſame with thoſe remaining of the laſt, each to each, or with the remaining ratio of the laſt.

$$\boxed{\begin{array}{ccc} & h,\ k,\ l & \\ A:B; & C:D;\ E,F & \\ G:H;\ K:L; & M:N;\ O,P;\ Q,R & S,T,V,X \\ e,\ f,\ g & m,\ n,\ o,\ p & Y,\ Z,\ a,\ b,\ c,\ d. \end{array}}$$

$$\left.\begin{array}{c} A:B,\ C:D,\ E:F\\ G:H,\ K:L,\ M:N,\ O:P,\ Q:R\end{array}\right\} \begin{array}{l}A:B=S:T\\C:D=T:V\\E:F=V:X\end{array} \left\{\begin{array}{l} G:H=y:z\\K:L=z:a\\M:N=a:b\\O:P=b:c\\Q:R=c:d\end{array}\right\}$$

$$\begin{array}{l} S:x=y:d\\ A:B=e:g(=S:T)\\ h:l=\overline{C:D}+\overline{E:F}\\ m:p=\overline{M:N}+\overline{O:P}+\overline{Q:R}\end{array} \left\{\begin{array}{c}\text{Then}\\ h:l=m:p.\end{array}\right.$$

Def. X.

$$S:X=\overline{S:T}+\overline{T:V}+V:X=\overline{A:B}+\overline{C:D}+\overline{E:F},\text{ Hyp.}$$

$$y:d=\overline{y:z}+\overline{z:a}+\overline{a:b}+\overline{b:c}+\overline{c:d}=\overline{G:H}+\overline{K:L}+\overline{M:N}+\overline{O:P}+\overline{Q:R},\text{hyp.}$$

$$e:g=\overline{e:f}+\overline{f:g}(=\overline{A:B}=S:T,\text{ Hyp.})=\overline{G:H}+\overline{K:L},\text{ Hyp.}$$

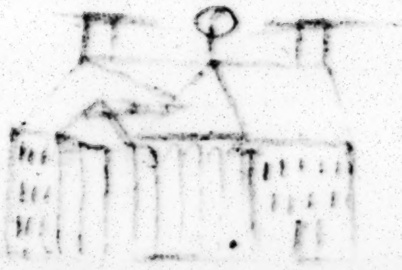
$$h:l=\overline{h:k}+\overline{k:l}=\overline{C:D}+\overline{E:F},\text{ Hyp.}$$

$$m:p=\overline{m:n}+\overline{n:o}+\overline{o:p}=\overline{M:N}+\overline{O:P}+\overline{Q:R},\text{Hyp.}=\overline{a:b}+\overline{b:c}+\overline{c:d},\text{Hyp.}$$

$$\text{Hyp.}\left\{\begin{array}{l}
\left.\begin{array}{l} e:f(=G:H)=y:z\\ f:g(=K:L)=z:a\end{array}\right\} \begin{array}{c}\because \text{Prop. XXII.}\\ e:g=y:a\end{array} \left\{\begin{array}{l} A:B(=S:T)=e:g,\ \text{Hyp.}\\ \because S:T=y:a,\ \text{Prop. XI.}\end{array}\right.\\
\left.\begin{array}{l} h:k(=C:D)=T:V\\ k:l(=E:F)=V:X\\ \quad\;\; — \;\; — \;\; —\end{array}\right\} \begin{array}{c}\because \text{Prop. XXII.}\\ h:l=T:X\\ \\ \end{array} \left\{\begin{array}{l} T:S=a:y,\ \text{Prop. B.}\\ S:X=y:d,\ \text{Hyp.}\\ \because T:X=a:d,\ \text{Pr. XXII.}\end{array}\right.\\
m:p(=a:b+b:c+c:d)=a:d,\ \text{Prop. XXII.}
\end{array}\right.$$

$$\left\{\begin{array}{l} h:l=T:X\\ T:X=a:d\\ a:d=m:p\end{array}\right\} \begin{array}{c}\because \text{Prop. XI.}\\ h:l=m:p.\end{array}$$

FINIS.



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ERRATA.

Preface, line 3, for Commendine's, read *Commandine's*.
Page 6, line 12, dele *removed*.
—— 14, line laſt but one, for $A+B:C$, read $\overline{A+B}:C$.
—— 15, line 7, for $n-1.C$, read $\overline{n-1}.C$.
—— ib. line 24, for equimultiple, read *multiple*.
—— 16, line 32, for onſequently, read *conſequently*.
—— ib. line 34, for $:B$, read $A:B$.